## Letter to the editor

The article by Norman Grant 'Radioactivity in the History of the Earth' in the Newsletter Articles Supplement, September 1996, under the sub-heading 'Rudolf Steiner & Radioactivity', quotes from notes of a lecture (5 October 1905) in the course 'Foundations of Esotericism' as follows:

> “In earlier times atoms were progressively hardening; now, however, they are coming more and more apart. Previously, there was no radioactivity. It has existed only for a few thousand years, because atoms are now splitting up increasingly".

I have checked this quotation with the original German (G.A. 93a, 3rd. edition 1987) It appears to be incomplete and inaccurate. My translation is as follows:

> In earlier times these atoms have more and more consolidated (*verfestigt*); now , however, they are moving (*treten*) more and more apart. Previously, radioactivity did not exist, therefore, one could not discover it earlier. It appeared first some thousands (*EINIGEN JAHRTAUSENDEN*) of years ago, because now the atoms are splitting up more and more.

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# Basic Gestures of Human Embryological Development

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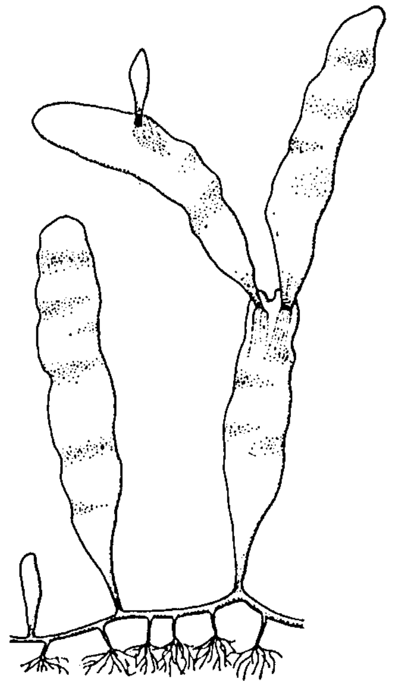
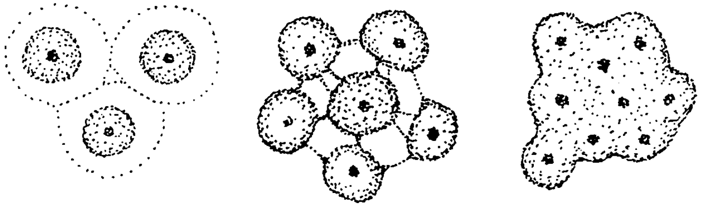
*Translated by Timothy Cox from’DieVorgeburtlichkeitdes Menschen - Der Entwicklungsgedanke in der Embryologie’ (Human Prenatality - The Idea of Evolution in Embryology, Verlag Urachhaus, 1982), Chapter 2*

In the beginning, there is the union of the egg cell with a sperm cell. Already in this statement, there is not only an observation which was first made for humanity under a microscope in 1944, but also a sweeping principle: the concept of the cell. In this view, a great variety of different structures have come to be considered as cells, since 1839 when Theodor Schwann proclaimed that all plants and animals are made up of them. A cell is any structure which has cytoplasm, a nucleus, and a membrane which demarcates the whole. The *general cell theory* (i.e. that all organisms are comprised of cells), which was proposed by Schwann and which is still advocated in textbooks to this very day, is untenable—in the cases of plants, animals, and the human being. Very commonly, tissues consist of multinucleate structures whose individual nuclear-cytoplasmic regions are not bounded by membranes. Such "plasmodia" comprise for example the fibres of the entire transverse voluntary musculature of the human body, and that is a significant volume. There exist very highly developed sea algae, for example the siphon algae (*Caulerpaceae*), which are comprised entirely of a single conterminous plasmodium with several hundred thousand nuclei. Upon closer inspection, it is not the case that there is a complete separation between the cells in even normal so-called cellular tissue, or otherwise their close co-operation would be impossible. Between them, there are countless cytoplasmic bridges (plasmodesmata, see fig. 9), which in fact not only functionally, but also morphologically, spring beyond the defined cell principle. A sheer aggregate of independent cells living next to each other would not yield an organism, but only a "colony."

Only the inclination toward atomic



8 The siphon algae *Caulerpaceae* becomes as big as a hand and despite all branching consists of a single contiguous cytoplasm with countless nuclei (After Schenk.)



9 Multinucleate structures. Left: Colony of single cells, living independently of each other, which are here externally held together by dead, excreted gelatinous substance. Middle: Tissue of "cells" which stay connected through cytoplasmic bridges (plasmodesmata). Right: Plasmodium (formerly also called syncytium) consisting of a single common cytoplasm with multiple nuclei

thinking, which would like to build life up additively out of individual building stones, gives the cell theory any right to survive, not the reality. An unconsidered, unconscious wish, not only to understand the biological world additively, but also to re-create it in this way and so to be able to manipulate it artificially, was the father of this thought. However, observation shows that all organisms which are composed of true tissues organize these tissues in a state between that of sheer cell colonies and of contiguous plasmodia, a state which allows the cytoplasm associated with a particular nucleus not only to collect around it but also to maintain a connection to neighbouring nuclear-cytoplasmic regions. This thoroughly rhythmical intricate construction can tend on the one hand toward the extreme of plasmodium (e.g. most of our musculature), or on the other hand toward the extreme of single cells (e. g. gametes). In the first case, after nuclear division, no cytoplasmic division takes place, but in the second case, the complete separation of cytoplasm occurs.

Therefore a general cell theory of organisms is untenable, unless we take "cell" to mean what the astute botanist Julius Sachs described as the "energide" in 1892: the functionally close connection between a nucleus and its directly adjacent cytoplasm—regardless of whether this is morphologically wholly, partly, or completely divided off from the neighbouring energide. In fact cells are spoken about today in the sense of energides, inexact as that may be. Here we must again learn to think the living connections which nature reveals, in order to get away from "thinking with bricks."

A further important observation was made by Richard Hertwig in 1884: he noticed that in most living tissues the size of the cell nucleus remains in constant proportion to its associated cytoplasm, this constant varying only according to the species. A larger cytoplasm will have a larger nucleus; a smaller cytoplasm a smaller nucleus (law of the species-specific ratio of cytoplasmic to nuclear volume). Now at the formation of the sex gametes, where the organism always, without exception, reproduces the unicellular state, the constancy of the nuclear-cytoplasmic relation conspicuously fails. The ovum becomes an ovum through the very process of forming so much cy-



counterspace. This asks that, in passing from the form of the skull to the form of the limb bone, one steps out of the region of finished forms, at least for a while, and into the region of movements, forces and activities. So the geometer has to become imaginative. Does that mean that he has to go into the domain of algebra which, with its deeper rhythm of number, embracing the imaginary numbers, determines solutions which cannot he drawn on paper? If so, then he would have to return again to 'real' geometry, for the limb bone is as hard as the skull.

**References**

1 Rudolf Steiner, *The Study of Man*. In German: Allgemeine Menschenkunde als Grundlage der Pädagogik (GA 293). 14 lectures: Stuttgart, 21 August to 5 September 1910. Translated by D Harwood and H Fox. Rudolf Steiner Press (1981), lecture 10, p 137.

2 Rudolf Steiner, *The Fifth Gospel* (GA 148). 7 lectures, Oslo, 1 to 6 October and Cologne, 17 &18 December 1913. Revised translation by C Davy & D S Osmond. Rudolf Steiner Press (1978), lecture 3, esp. pp 58-61. *Aus der Akasha-Forschung, Das fünfte Evangelium.*

3 Rudolf Steiner & Ita Wegman, *Fundamentals of Therapy. An Extension of the Art of Healing through Spiritual Knowledge* (GA 27). Translated by E A Frommer & J M Josephson. Rudolf Steiner Press (1983), chapter 1, pp1-6. *Grundlegendes für eine Erweiterung der Heilkunst nach geisteswissenschaflichen Erkenntnissen*.

4 Rudolf Steiner, *Knowledge of the Higher Worlds* (GA 10). Revised translation by D S Osmond & C Davy. Rudolf Steiner Press (1976). Under 'Preparation', pp 46-49. *Wie erlangt man Erkenntnisse der höheren Welten?*

5 Rudolf Steiner, *Knowledge of the Higher Worlds* (GA 10). As ref. 4 but under 'Control of Thoughts and Feelings', pp 63-67.

6 Euclid, *The Elements*, Book 1, Propositions 27-29. Translated into German by Heiberg and thence to English by T L Heath. Dover, New York, (1956). Vol. I, pp 307-314.

7 Rudolf Steiner, *The Philosophy of Freedom.* Translated by Michael Wilson. Rudolf Steiner Press (1964), chapter 5, 'The Act of Knowing', Author's addition 1918, pp 78-81. *Die Philosophie der Freiheit.*

8 Rudolf Steiner, *Truth and Science* (sometimes entitled *Truth and Knowledge*). German title: *Wahrheit and Wissenschaft*. Translated by R Stebbing. Steinerbooks USA (1981). Chapter VIII, 'Practical Conclusion'.

9 Rudolf Steiner, *The relation of the different branches of natural science to astronomy.* (GA 323), 18 Lectures, Stuutgart, 1-18 January 1921. Lecture 1. Translated by George Adams, typescript Rudolf Steiner House Library, London. In German: *Das Verhältnis der verschiedenen naturwissenschaftlichen Gebiete zur Astronomie.*

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of feeling, that the universe is not really unending, as the word ' infinity' implies, but is embraced by a father-like counter-centre from which all destiny works. This same feeling of consolation rises upon the soul-horizon whenever one passes from the traditional geometry over to exercises in projective geometry, which require that the geometer, with pencil in hand, repeatedly takes conscious account of the plane at infinity or 'celestial plane' as George Adams has referred to it.

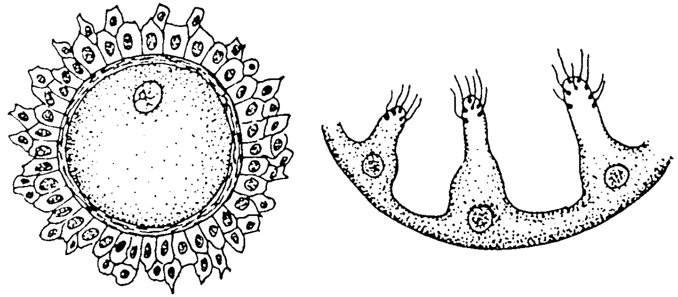
An experience also related by some people is the following: In looking back methodically over one's life, noting especially what has actually been done with the limbs, then a pattern emerges. A certain tendency to act in one way may be seen as a series of repeated episodes, the repetition of which has not been fully conscious hitherto, but the whole story of the movement and gesture with which one has lived might be seen as a theme in the biography. The remarkable thing is that, when placed thus in perspective, the causes behind one's deeds become clear, and these are found to be *ideas*. Rather they are ideals, for they have worked into our actions. In the last chapter of his book: *Truth and Science,*[8] Rudolf Steiner has set out explicitly what in practice it means to awaken to the causes standing behind one's actions. Two sentences from this chapter run: 'In its whole character, our conduct of life is determined by our moral ideals; these are the ideas we have concerning our tasks in life or, in other words, the ideas which we form of what we would accomplish through our deeds'. He goes on there to describe how it is that, so long as we remain unconscious of the ideas which work in our actions, then those ideas nevertheless work, compelling us to act, puppet-like. Only when we have penetrated to the ideas at work in our acting do they then work in us in a way which allows us to rule our deeds. This knowledge-activity, he thus shows, is the path towards freedom.

The point here relevant is that in seeking the origin of the actions of our limbs we find, lo and behold, ideas! This seems to accord with the fact that the core of the limb system is the world-periphery plane and also with the observation, noted above, that our inner experience of concepts and ideas is akin to our sense experience of very distant objects.

**Conclusion**

I have sought to draw attention to experiences, accessible via observations which are readily made, which speak about man's connection to the world through his limbs. The foregoing cannot be considered a demonstration that the limbs are centred in the world periphery, but it is intended to be a contribution towards such a demonstration, which it is surely the business of sculptors, geometers, teachers and others to work upon. Whatever approach one takes to the skull-limb problem, the medium in which one works will, I suppose, tend to set its own limits, be it the clay, the mental pictures of geometry or the practical situation with which one grapples. Coming from the tidy realm of the geometer, I am always happy when there is form and this one-sidedness may be at times a hindrance. For, as the foregoing is intended to show, the limb-sphere is not picture-like and cannot be grasped in the usual way, with the mental picturing activity connected with the skull. The limb-sphere is seed-like, all potential and is experienced consciously only with the help of the imaginative idea of



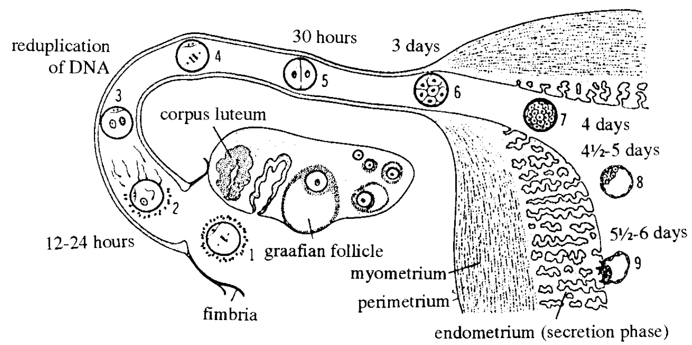


10 Left: Ripe human egg cell after ovulation, surrounded by the narrow zona pellucida (a transparent gelatinous envelope) and many low-cytoplasm nurse cells, together forming the corona radiata which had taken part in the maturation of the egg cell. Right: Sertoli plasmodium from the lining of the seminiferous tubules, in whose cytoplasmic projections the sperm cells reach maturity.

toplasm that it cannot maintain the normal relation with the (actually relatively large) ovule nucleus (polar body). The sperm cells on the other hand reduce their share of cytoplasm so much that it comes to surround the cell nucleus only as an extremely thin hull and braids itself together to form the flagellum.

In the case of the female cell, the nuclear-cytoplasmic proportion polarizes toward the cytoplasmic; while the spermatozoa polarize toward the nuclear. In 1841, this characteristic of the sperm was first noticed by Kölliker, and of the ovum by Remak. When Lothar Vogel therefore sees the loss of cell nature in mature sex gametes, it is an overstatement, since all constitutional parts of a cell remain present. And yet, the harmonic co-ordination of organelles inherent in normal living tissue has been lost in both types of cell.

It is even more telling to note how the organism can form such polarization in the first place. Namely, the sperm cells can only ripen in plasmodium, in the so-




11 Early development and path of the human germ cell in the first week from ovulation (1) until implantation (9). (From Langman).

called Sertoli "cells," which meanwhile become conspicuously rich in cytoplasm, while helping the many sperm cells reach maturity in a cytoplasmic process. The egg cell on the other hand, develops with the help of the cells surrounding it, the "corona radiata," which is made up of large numbers of nuclei with little cytoplasm. The nurse cells corresponding to each type of germ cell complement what the ripening sex cells themselves are lacking! But in this way, the human gametes can polarize themselves so extremely that a sperm cell represents only 1 / 100 000 of the volume of the egg cell.

At conception, a biologically exceptional process occurs. Every higher organism (as is generally known from the difficulties involved with organ transplantation) rejects and destroys every kind of foreign protein through immune reactions. Yet even if they originate in different individuals, the living chemistry between egg and sperm cells do not mutually exclude each other. This means that the individualized albumin structure appropriate to the mature organism does not yet exist: it has actually been suspended with the breakdown of the ratio of cytoplasmic to nuclear volume. Rudolf Steiner often spoke of a "chaotification" of the germ cell albumen (GA 205, 226). The direct expression of this is the possibility of foreign insemination.

At the moment of conception, the reception of the sperm nucleus mitigates the relatively short supply of nuclear substance in the ovum. In the process of cell division which commences immediately, the cytoplasm, still by far the greatest proportion, now becomes distributed among the quickly multiplying nuclei, but the volume of cytoplasm does not increase. This so-called cleavage process has the effect of re-establishing the normal ratio of cytoplasmic to nuclear volume. By experimentally changing the quantity of cytoplasm in fertilized animal eggs (e.g. of sea urchins) through the removal of egg fragments, fewer nuclei and thereby fewer cells form: the cell count is related to the ultimate quantity of egg cytoplasm (Seidel 1953). After restoration of the ratio of cytoplasmic to nuclear volume, the further development proceeds according to the tight interplay of both active polarities. From now on, it is possible to speak of ectodermic tissue capable of organization. The cytoplasm lives in a temporal rhythm and becomes the space in which the etheric can intervene. The nuclei convey the supertemporal species-specific order and all astral access—presuming it is already allowed to bring in the anthroposophical terminology (q.v. Chapter 3).

The first cellular structuring of the developing organism is therefore—and this must be pointed out with all its consequences—not an accretive *addition* of cells, but rather a *division*, beginning with the unity of the fertilized ovum and leading to the multiplicity of twelve or sixteen individual blastomeres which constantly remain in contact with each other. This uniformly large mulberry, the morula, passes along the uterine tube to the uterus, moving forwards by means of the beating of the ciliated epithelium of the inner lining of the uterine tube. Thereby, through the release of transmitters, there is already a fully fledged mutual contact with the maternal organism along the humoral pathway. Thus the travelling zygote is affected by the ovarian corpus luteum hormones and the corpus luteum is conversely influenced by secretions of the morula. (Seidel 1968).

With the loss of the zona pellucida on approximately the fifth day of develop-



just in front of us. Hence the camera lens has the symbol ‘¥‘ written upon it.

Geometry texts since those of the Greek geometers have used diagrams such as the following as part of proofs[6] (Fig. 3).

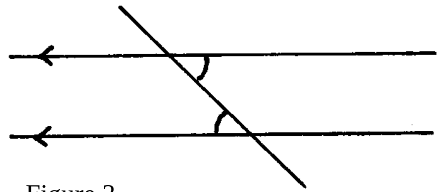


Figure 3.

The two marked angles are said to be equal because this is an instance of the alternate angles generated by a line crossing two parallel lines. No reader of this proposition, however, can claim to experience the parallelism of the two lines, either with the eye or in the activity of mental picturing. All views of the diagram, including the view normal to the plane of the diagram, render the two lines convergent and therefore concurrent in a point. Attempts to picture the diagram mentally are subject to the same restriction. The actual experience of the 'parallelness' does not lie within the experience of the mind's eye.

Parallelism *is* experienced somewhere in the whole range of our activity, however. Looking at the matter reveals that it is experienced in thinking, specifically as a concept. The diagram is a particular image, a perceptible image, of the concept of parallelism, necessarily imperfectly rendered. Only if the plane of the diagram could be removed to the plane at infinity would the eye's experience of the two lines tend towards and finally become an experience of parallel lines. Only then would the effect of perspective be completely flattened-out; no section of the lines would be any further away than any other. Of course this same perspective would determine the two lines so apparently close together as to be no longer resolvable from one another. Nevertheless it may be good to follow through such experiments, as far as the eye can do them and then further as a mental exercise, for the following reason: that in his book *The Philosophy of Freedom*[7] Rudolf Steiner draws our attention to the fact that for pure thinking naïve realism is true; concepts and ideas are as they seem. Thus is shown the connection between our visual perception of what is far away and our inner experience of concepts.

A third way in which we experience the plane at infinity is that which flows literally through the bodily limbs. When reflecting afterwards upon some unusual event which has involved us, our hands and feet, it may happen that we become aware of the fact that many separate factors have worked together in a way which chance would assess highly unlikely. We become aware that the event was brought about by a process working from whole to part. Only thus could the seemingly rare constellation of factors be formed, at the particular time and place, without putting the whole world off its axis. It is, after all, destiny and works from the surroundings, streaming into the body via the limbs, in a manner of which we are usually initially unconscious. Perhaps trust in the working of destiny is already common, perhaps among children and among adults in a similar way, but what an awakening help it is to illumine this with the imaginative idea of the world periphery as the centre of counterspace in which is rooted the world limb system, working in a direction polar to the forces of the head. No less than this boon to cognition is the consequent consolation to the life

**Picture and seed**

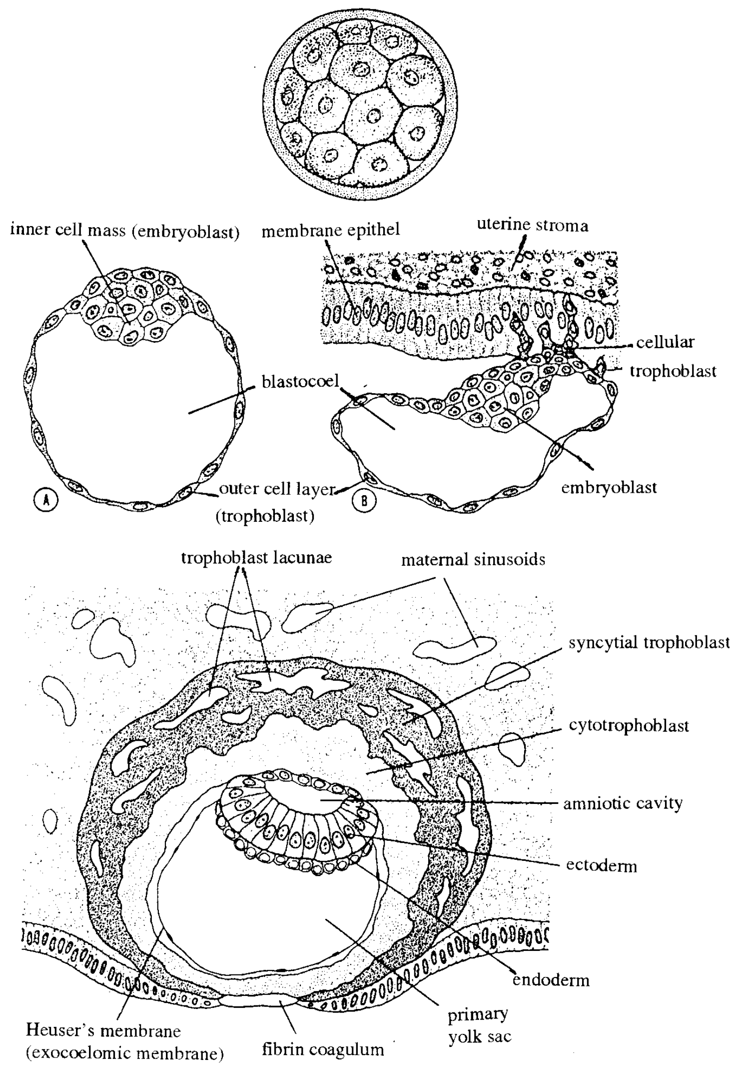
Thus far is the imaginative idea of Space and Counterspace helpful in portraying the essence of head and limb. They are seen to be polar opposites. The same polarity occurs in nature, perhaps in many ways, one of which is the polarity between fruit and seed. The fruit-form is akin to the head process; it is filled with physical substance swelling out towards the surroundings. The flesh of the fruit is the distillation of the foregoing growth of the plant. We even use the word 'fruit' to mean result or product. The seed-form is akin to the limb process; it is also filled but in a manner opposite to that of the fruit. The content of the seed lies, from the viewpoint of the bodily senses, all around it. The living body of the point-like seed, that which is filled with etheric substance, lives around it. This invisible substance is potentially destined to pour itself into visible form in the future, after the seed has germinated, and so to build a future plant. How the future plant will be is something already there, hidden from the bodily senses, within the etheric or counter part of the seed. Biodynamic growing, beyond what organic growing already recognises, has to reckon with these forces which are counter to the physical ones, streaming in from the world periphery.

The fruit is a finished picture; it is an outwardly manifest thing which has come to completion. Its form and substance are retrospective. The seed is all potential; it is the as yet unstarted plant, forward oriented.[5] Therein is shown the idea of Picture and Seed, and its kinship to the polarity of Skull and Limb. The entire *Menschenkunde* lecture course,[1] given to the teachers-to-be at Stuttgart, seems to carry this theme, namely that the head is picture like, the limb-system is seed-like. The second lecture appears crucial in this respect. Therein Rudolf Steiner describes these two poles as two streams working in each human being. The first stream, which the soul experiences as its activity of mental picturing, flows from before and through the gate of birth. The second stream, called 'Will/Seed/Imagination,' flows from the present, upon Earth, and reaches forward through the gate of death and beyond. Perhaps Rudolf Steiner considered the grasping of this idea in a practical way something essential for the group of those who were to become school teachers.

**Our experiences of the world periphery**

The phenomenon of parallax, by means of which the distance of distant objects is measured, bears witness about the plane at infinity. As the distance increases, the parallax gradually disappears. It is a 'limit' process: the limit, the plane at infinity, is never experienced by the bodily senses but the process towards the limit is. The sense experience of this process implies that an object in the plane at infinity will be identical for all observers, from all viewpoints. The *appearance* of objects in the plane at infinity must then be non-existent. Only their absolute being, their existence, would be experienced. If the senses could reach to the world periphery, then naïve realism must be true for their experiences of what is there: one could say about them that things are as they seem. When observing the stars, the subjective element seems to vanish. Of all objects, the clear night sky is the one which commonly calls forth a naïve childlike feeling in which there can be wonder. The eye brings what is infinitely far, near. The lens, as in man-made instruments, produces an image, even of the stars,






12 Early embryonic stages.

*Top:* Human morula on the fourth day, still surrounded by the zona pellucida, which later disappears.
*Middle:* A. Human blastula (blastocyst), about 4½ days old; B. Beginning of the implantation of the blastula of a rhesus monkey.
*Bottom:* Human blastula, about nine days old, after successful implantation in the uterine mucous membrane. The embryoblast has differentiated into ectoderm (with first amniotic cavity) and endoderm. (From Langman).

ment, a second structuring occurs: the fluid-filled spaces between the blastomeres flow together to form a cell-free cavity, turning the morula into a blastula. Thus nearly all the blastomeres collect together into a strong, enlarged surface, the trophoblast (ectodermic nutrient layer) and the embryo begins to expand. Simultaneously however, it keeps a small portion of cells compactly together, the embryoblast (inner cell mass). But in fact this does not transform into the actual embryo, but for the time being merely into further inner cavities: the amnion and the yolk sac. Research on mammalian embryos has shown that the trophoblast and the embryoblast cannot continue to develop independently of each other; they mutually depend on one another for their existence (Petzoldt).

Toward the end of the first week, on the sixth day, the blastula affixes itself to the upper inner lining of the uterus and becomes covered by a mucous membrane. With implantation (nidation), an intensified contact with the maternal organism begins through direct tissue adjacency. Theoretically, due to the genetic make-up of the sperm cell, the mother’s body should reject this foreign embryo through a violent immune reaction, just like with every other foreign protein. This actually does occur in rare cases such as immunologically instigated sterility. But in general, the immunological defence system of the maternal organism is obviously inhibited, or otherwise the development and functioning of the placenta, which binds mother and child over a period of many months, would be impossible. Furthermore it is important that the implantation normally succeeds in the upper part of the uterus, not according to gravity. If it were to implant lower at the cervix, the placenta previa would develop blocking the birth canal, located so as to be mortally dangerous at birth if not surgically ameliorated.

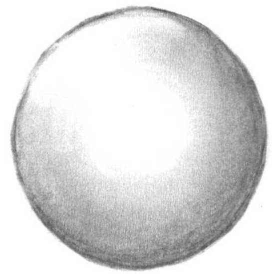
As already mentioned, after implantation further cavities form in the embryoblast of the blastula: the amniotic cavity (amnion), in which the future embryo and foetus will float weightlessly in the amniotic fluid, and the yolk sac which will function as the first location of blood formation and also as an embryonic hormonal gland. The decision about whether the embryo will become identical twins, triplets, etc., is made by about the twelfth day, depending on whether the dividing blastomeres of the embryo form multiple embryoblasts or yolk sacs. For human beings, it is known that up to monozygotic quintuplets can develop (Starck). Today increasingly more nonidentical (fraternal) twins, triplets, etc., are being born, where treatments with ovulation stimulating hormones are undertaken or when polyovulation occurs in compensation for residues left by ovulation inhibitors. The frequency of twins is about 1% of all births, and of these only 20% are identical.

Up to the twelfth day, it is not only decided whether the embryo will develop into a single individual or into multiple individuals, but also whether the actual physical development occurs at all. If not, the embryo remains as a blastocyst, which shortly dies; the substantiated estimations of embryologists seem to indicate that about half of all embryos atrophy in the early stages of their development and are expelled unnoticed with menstruation (Langman). However if multiple yolk sac complexes do form at the amnion, identical twins, etc., will develop. Toward the end of the second week, the embryo finally becomes determined as an independently living organization.

But now, from where does the physical development actually proceed? Precisely at the still extremely tiny surface of contact between the amnion and the yolk sac. The amnion portion will become the initial outer skin (the ectoderm) and the upper skin, nervous system, and part of the sense organs will arise from it. The yolk sac will become the "inner skin" (the endoderm) which will develop into the entire digestive tract with all its auxiliary organs (liver, gallbladder, pancreas). During the first half of the third week, the middle germ layer, the mesoderm, works its way



Figure 1



The limb sphere centre, as Rudolf Steiner has described it,[1] is not perceptible to the senses. It cannot even be grasped as a mental picture, because mental pictures borrow their contents from the experiences of the bodily senses. This centre lies in the whole world periphery. The radii of the limb sphere reach inward from the periphery towards a point in space, yet they are not seen fully to reach a point. The physical limbs are the last traces of radii of the limb-sphere which apparently do not, on their journey inward, reach a point. The limb-sphere surface is thus the surface of an inside-out sphere (Fig 2).

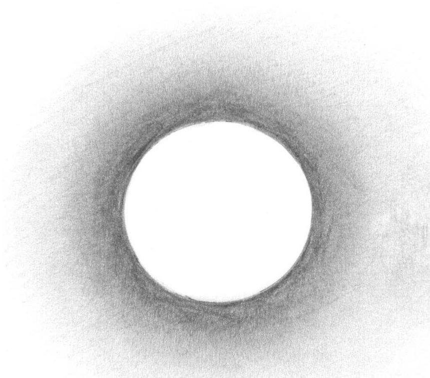


Figure 2

In this way is the following idea made precise: that whereas each individual human being living upon the Earth has his or her own head, centred within the body, we all together, on the other hand, as one humanity partake of the one universal limb system in so far as the limb system of each human being is centred in the archetypal plane, the plane at infinity. This world-periphery-plane is the core of the etheric world. From there originate the forces of life.[3] The head reaches out from its particular physical centre toward the one universal plane: at infinity. The limb system, from its universal etheric centre, the plane at infinity, reaches inward towards a point, or points, in the body.

Essential to this experience of the two poles of physical space and etheric, or counter, space must be the exercises of observing phenomena of blossoming and of dying, described by Rudolf Steiner in his book: *Knowledge of the Higher Worlds*.[4] What springs from the head and flows down through the body, as does the development of the young child up to about 7 years, is a Spring-like process of blossoming. It goes from the pole of death towards the pole of life. On the other hand what streams from the periphery, through the limbs and towards the head, as do the bodily processes of dying which accompany the development of the consciousness soul between the ages of about 35 and 42, is an Autumnal process flowing from the pole of life towards the pole of death.

That each of us has his or her own head centre is in keeping with the fact that in this sphere man can attain freedom. With the limbs the opposite is the case: although physically we each appear to have our own limbs, the fact is that everyone's limbs are rooted in the one universal limb-system, the plane at infinity. To this common ground man is bound by world necessity. That we all partake of what is ultimately the one limb system accords with the fact that brotherhood too is part of man's condition.

# The polar relation between the human skull bones and limb bones

A development, using the imagination of counterspace, of Rudolf Steiner's indications given during the tenth lecture of his *Menschenkunde* course[1]

Gordon Woolard

**The three spheres**

As Rudolf Steiner has described in the above-mentioned lecture,[1] each of the three parts of the human body belongs to its own sphere. His use of the term 'sphere' is wider than, but equally precise as, its common use in geometry. The physical skull reveals almost the entire surface of the head sphere. Although showing physically only a part, the bodily trunk nevertheless implies the whole surface of its trunk-sphere. In the case of the limbs, the matter is different because the limbs are not part of the surface of their sphere. They are parts of radii of their limb-sphere.

Putting aside, at first, the inside-out character of the limb-sphere, it proves helpful to attend solely to the fact that the limbs are traces of radii and not surface. This distinction cautions us to beware of comparing, however precisely, the forms of head, trunk and limb *per se*. Rudolf Steiner's portrayal, both in his words and his board sketch,[1] of the three whole spheres anyway steers us away from what is experienced by the physical senses alone. He seems to point us towards a comparing of the three whole spheres. These spheres are experienced in the following way: The physical senses suffice for the head sphere. The trunk-sphere can be visualised or mentally pictured. To experience the limb-sphere these are not enough - we have to be active in imagination.

During the said lecture[1] Rudolf Steiner indicated that a metamorphic relation does nevertheless exist between the forms of the shell-shaped head bones and the limb bones: 'To do this you have to adopt a certain procedure. You have to deal with the tubular bone of the arm or the leg as you do with a glove or stocking when you turn it inside out to put it on.' During the 'Astronomy Course'[9] he urged mathematicians to work upon this problem. It is widely felt that the secret of this metamorphosis must be fundamental not only to anthroposophy but also to Christianity, in that it seems essential to what anthroposophy has to say about the Mystery of Golgotha.[2]

Therefore we are justified in seeking to work out this path of metamorphosis as many have sought to do. What, however, I would stress here is that it may be fruitful to compare the three spheres before going ahead to any particular detail, for the reason that in so doing we undertake a certain step in inner activity which may otherwise he missed; namely the step from mentally picturing to imagination.

**Sphere and countersphere**

Being perceptible to the bodily senses, the head sphere surface is easily pictured. Its centre is a point in space and its radii reach outward toward the periphery of space. They do not arrive there, however, because they stop at the surface which encloses the physical substantiality of the head ( Fig. 1).

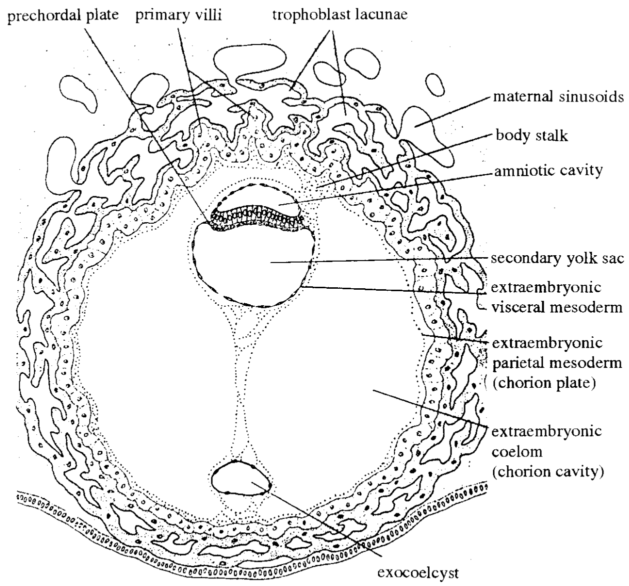


from the outer epiblast of the embryo (from the trophoblast), over the body stalk, to form a layer between the ectoderm and the endoderm. At the same time however, the cells of the middle layer ingress from the posterior ectoderm (out of the primitive streak) and later out of the anterior ectoderm (mesectoderm of the neural crest) into the mesoderm. This layer will become the most important organs of the rhythmical system: blood circulation, heart, musculature, bone system, kidneys, and genital organization. The lungs likewise arise as an "intermediate" formation out of the transitional region between the pharyngeal ectoderm and the upper esophageal endoderm.

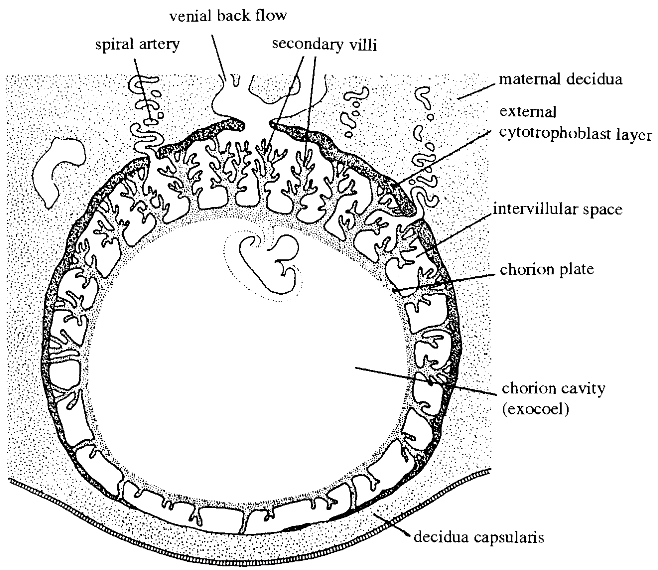
Here a strikingly noticeable step of development is the transition from the flat embryonic shield (embryonic disc) to the space-filling shape of the developing body. This happens on about the 18th day. Observe "that exactly in the normal early development period, numerous individual differences and variations appear." (Starck, pp. 221, 229), so that some embryos were at first considered to be abnormal, until it was noticed that the range of individual fluctuations could differ by days. (With rhesus monkeys however, it was easy to devise a definitive chronological table of norms for embryological development; Heuser and Streeter).

Compared to this organogenesis, the surface of the trophoblast grows more quickly and becomes covered by villi on all sides, becoming a *chorion*. At about the third month, this same chorion will differentiate and transform into a *serosa* (without chorionic villi) and a villous disc-shaped placenta. But before that, in the first and second months, the embryo is really just a small inner appendage to a very much larger blastula. Initially, the formation and development of the blastocyst overshadows the formation of the embryo. It must be emphasized repeatedly, that not only the much smaller embryo but also the blastula is composed of infantile tissue. What is the meaning of the formation of this all-encompassing peripheral hollow organ, which does more for the life of the germ than the embryo itself?

The natural science of anthropology considers the placental membranes to be mere "ancillary organs," which serve the "adaptation" into the intrauterine milieu. It sees only organs which assist the embryo in its development. Anthroposophical spiritual science, however, opens up a much wider perspective, obligatory in order to clarify the reality and effectiveness of the characteristic individuality, of the 'I'. In a lecture at the Philosophical Congress in Bologna in 1911, Rudolf Steiner described the nature of the human 'I'. It lives in the thinking contemplation of the world, aware of itself most clearly. Rudolf Steiner explains that the 'I' is a part of the surrounding world, not a member of the physical body: he uses the example of mathematical thinking, an inner activity which is conducted by consciousness in a self-imposed logically consistent way independent of the sense world, which can derive laws about the physical world and which can also discover these same laws in this physical world, without them needing to be modified. The body is only a mirror, which helps the 'I', living and being in the world, gain consciousness of itself and of the lawfulness of the world. "… a better mental picture of the 'I' can be reached epistemologically, when it is not pictured as being located within the physi-




13 Embryo at 13 days. The nutrient layer of the embryo (trophoblast) has developed on all sides to take in the nutrients available from the maternal sinusoids. In addition to the primary yolk sac (exocoelcyst), a new, secondary yolk sac has formed. (From Langman).




14 The spherical germ with the small embryo at about 30 days. The formation of chorionic villi is fully underway (actual diameter of the blastocyte about 3 cm). (From Langman).

cal organization and the impressions coming to it from outside, but rather when the 'I' itself is considered to be within the lawfulness of the things themselves, when the physical organization is seen as being only something like a mirror, serving to reflect the weaving of the 'I' lying outside of the body in the transcendental back to



Litteratur' 1883-1897, 5 Volumes, reprinted Dornach 1975, in the Complete Edition Bibl. Nos. 1a-e. In Volume II, p.45f. we find: 'I hear that I am accused of being an adversary, an enemy of mathematics in general. However no one values mathematics higher than myself, for it accomplishes exactly that which is wholly denied to myself to achieve. Concerning this I would gladly clarify my position and choose to this end a single means- the words and lectures of other significant and notable men. What concerns the mathematical sciences must by no means impress us through their nature or multitude. They owe their certainty primarily to the simplicity of their objects.'

9. *Even there, where we require no calculation*; Goethe 'The Experiment as Mediator between Subject and Object'. ibid, p.19.

10. *Letter to Herder* : 17th May 1787, in 'Italian Journey.'

theosophische Kongress in Amsterdam' in 'Lucifer-Gnosis. Grundlegende Aufsätze zur Anthroposophie und Berichte aus den Zeitschriften 'Luzifer' und 'Lucifer-Gnosis' 1903-1908, GA Bibl. No. 34, Dornach 1960, pp.539 ff.

2. *Plato* : On the significance of mathematics, see- 'The Republic', Part 8[ Book 7], in Penguin Classics 1980, pp. 331-333. Translated by Desmond Lee. On page 332 we find : "Soldiers must study them so that they can organise their armies, and philosophers so that they can, as they must, escape from this transient world to reality." In the further course of Plato's dialogue between Socrates and Glaucon the talk is about whether arithmetic and counting should be compulsory subjects. In this connection the discussion continues : "'You see therefore,' I pointed out to him, 'that this study looks as if it were really necessary to us, since it so obviously compels the mind to use pure thought in order to get at the truth.'" (p. 333)

3. *There is only so much true science* : See I.Kant, 'Metaphysical Foundations of Natural Science' (1776). Preface. " I maintain, however, that in every specialised theory of nature one can only meet with so much *actual* science, as one can find *mathematics* therein."

4. *Newton and Leibniz* : Isaac Newton developed certain fundamental thoughts on differential calculus for his own private use. G.W. Leibniz, who also occupied himself with differential calculus, had methodically developed it further, made it fruitful for general use and published on it earlier than Newton. Differential calculus makes possible calculations with differentials, that is, with infinitely minute differences, and is together with integral calculus of great significance in dealing with all problems of the exact Natural Sciences and Mechanics. Considered together, differential calculus and integral calculus are called Infinitesimal Calculus.

5. *Karl Friedrich Gauss* , 1777-1855. He discovered that the axiom system of classical Euclidean geometry is only one of many possible.

*Bernhard Riemann* , 1826-1866. His geometrical investigations were later brought together under the heading, 'Riemannian Geometry.'

*Oskar Simony* , 1852-1915. Professor of Mathematics at the College of Agriculture in Vienna. He wrote among others, "Gemeinverständliche, leicht kontrollierbare Lösung der Aufgabe: 'In ein ringförmig geschlossenes Band einen Knoten zu machen' und verwandter merkwürdiger Probleme"[Generally understandable and easily controllable solution to the problem: 'To make a knot in a ring-shaped closed ribbon,' and other curious related problems.] 3rd enlarged edition Vienna 1881. See also Rudolf Steiner- 'Towards Imagination' GA Bibl. No.169, Anthroposophic Press 1990, and 'Die vierte Dimension, Mathematik und Wirklichkeit' GA Bibl.No. 324a, Dornach 1995.

*Friedrich Jakob Kurt Geissler* , born 1859. Wrote among others, 'Die Grundsätze und das Wesen des Unendlichen in der Mathematik und Philosophie' [The Principles and Essence of Infinity in Mathematics and Philosophy] (1902), 'Grundgedanken der Übereuklidischen Geometrie' [Foundational Thoughts of Super-Euclidean Geometry] (1904)

6. *The birth of the higher Manas out of Kama-Manas* : Here Rudolf Steiner uses the customary Theosophical designations. Later he replaced these with Anthroposophical concepts. 'Kama-Manas' or 'lower-Manas' means earthly consciousness in contrast to 'higher-Manas' (Budhi-Manas), also called by Rudolf Steiner 'Intellectual soul'.

7. *Arupa-Realm in contrast to Rupa* : Arupa-Realm or Arupa-plane, also Arupa-Devachan, designates higher Devachan, that is the world of reason, the world of true Intuition in contrast to the Rupa-realm (or Rupa-Devachan, lower Devachan, that is the heavenly world.)

8. *Goethe* : See his essay, 'Über Mathematik und deren Missbrauch sowie das periodische Vorwalten einzelner wissenschaftlicher Zweige, [On mathematics and its misuse as well as the periodic prevalence of individual scientific branches], in Goethe's Natural Scientific writings, edited and commentated by Rudolf Steiner in Kürschner's 'Deutsche National-



the 'I' by means of the organic physical activity." (GA 35).

Two important concepts for human self-knowledge are thereby identified. First it becomes clear that the presence of the 'I' and the consciousness of the 'I' are by nature not identical. This can be confirmed by considering the everyday experience of waking up in the morning. One is absolutely certain of not having had any consciousness of the 'I' for several hours and yet one still recognizes one's own personality as being identical with that of the previous day. The unconsciousness of sleep does not interrupt the existential continuity of our 'I'-identity. Second it becomes clear that the usual *consciousness* of the 'I' can only be experienced by means of the body, especially by means of temporary cerebral activity, but this is not true for the *continuity* of the 'I' itself. The true, active 'I', as opposed to the merely reflected 'I', does not live in the body, but rather as a supersensible, unincarnated being living in the context of the world.

Many modern thinkers and poets have known—and many more presentimented—this double nature of the 'I', which on the one hand is body-centred, and on the other world-connected. Perhaps the first was Johannes Scheffler, known as the mystic Angelus Silesius, who, at the dawning of the modern era, wrote:

> I don't know what I am, I am not what I know:
> A thing and not a thing: a dot and yet a bow [sic].
>
> Ich weiß nicht, was ich bin, ich bin nicht, was ich weiß:
> Ein Ding und nicht ein Ding: ein Tüpfchen und ein Kreis [lit. "circle"].

Rudolf Steiner (GA 316) once drew the attention of the doctors who were asking him for advice to the higher, peripheral 'I' of each human being: this *spherical* human, which remains spiritual all during life, actually gains a physical body for a brief time—the extraembryonic membranes. They always form spherically. Through them, the higher human being actively partakes in the development of the embryo. By this means, the physical organ of this spiritually active spherical human dominates at the beginning. On the other hand, nothing develops as quickly, as soon, or as big in the physical body (which will later mirror the consciousness of the 'I') as the central nervous system, including its tightly folded cerebrum, which in the human being is especially powerful.

Only at the birth of the earthy human being does the body of the spherical human die—the afterbirth. From then on, a division begins in the human being between its higher and its body-bound existence, which are no longer a unity. This tension is what makes us into the searching and striving human beings that we are, in other words, into earthly human beings. Friedrich Schiller (in his letters *On the Aesthetic Education of the Human Being*) put this archetypal human experience in the following words: "One can say that every individual human being, due to inheritance and determination, bears a pure, ideal human being within, and the greatest task of human existence is to reconcile this unchangeable unity with all one's pursuits." And Friedrich Rückert versified:

> Before each of us stands a picture of what we shall become.
> As long as we don't reach that, our peace cannot be done.
>
> Vor jedem steht ein Bild dess', was er werden soll.
> Solang er das nicht hat, ist nicht sein Friede voll.

According to Rudolf Steiner, initially this higher human being is already present before birth in a (veiled) physical way, independent of the forces of heredity which mould the embryo.

> This body, which is formed according to heredity, is a model—the higher human uses it as a model. The higher human puts the earthly substance into this body. This earthly substance, which the human being takes into its body in the first seven years, would be formed completely differently if the higher human were to work only according to those forces which it brings from its pre-earthly existence. It would call a completely different form into being. At birth, it does not tend to form a being like the earthly human being, having eyes, ears, and a nose. Rather, it tends to form the human being in such a way, that the physical development is actually left rather neglected by this pre-earthly being. Instead, the greatest emphasis is placed on the remaining sheaths. Those parts which do not develop in the embryo are developed in the astral*, in the 'I'-organization. So one can say, looking at the physical embryo: this physical part of the embryo, this is certainly wonderfully formed, but in its development the pre-earthly human being plays only the smallest role. On the other hand, the human being, the pre-earthly human being, has the greatest share in everything which is going on all around the embryo. The pre-earthly human being is alive in those parts which eventually become physically exhausted, which will be thrown away, like the chorion, amnion and so on. In there, the pre-earthly human being lives.

Such a representation gives us a completely new outlook on the placental membranes, which are so easily disregarded as being unimportant. It also helps us understand anew the deep respect that those people still bound to the older forms of tribal spirituality attribute to the afterbirth. It is reported of the Iraku in East Africa that the husband of the wife who has given birth undergoes ritual offerings in order to "appease the afterbirth, so that it will not harm him." (Kohl-Larsen). And even today the Favelados, a peasant population in Brazil, handle the placenta very carefully: it is buried single-handedly by the mother with mythical protection images. (Craemer).

* by astral, the lower soul body is meant



spirit of mathematics, everywhere after the manner and model of mathematics. He says, '... even there, where we require no calculation, we should proceed as if we were accountable to the most stringent Geometrician. For the mathematical method, on account of its deliberation and purity, immediately reveals every jump in an assertion. Its proofs are nothing more than elaborate explanations, and those things which are brought together were already there in their simple parts and in their entire sequence; and further viewed in their entire range, found to be correct and irrefutable.'[9] Goethe seeks to grasp the qualitative in plant-formations using the exactness and clarity of the mathematical way of thinking. Just as one puts particular values into mathematical equations to understand the manifoldness of specific cases under a general formula, so Goethe sought the Primal-Plant (Urpflanze), which is all-embracing in the realms of the Qualitative and Spiritual-Reality. Regarding this he wrote to Herder in 1787 : " Further I must confide to you, that I am very close to the secret of plant generation and organisation, and it is the simplest thing imaginable..... The *Primal-Plant* will be the most wonderful creation in the world, for which Nature herself shall envy me. With this model and its key, it will be possible to invent plants into infinity, that consequently must exist, that is to say, even if they don't actually exist, could nevertheless."[10] Thus Goethe sought the totally formless Primal-Plant and strove to derive the plant formations from it, just as a mathematician derives specific forms of lines and planes from an equation. - Goethe's way of thinking in this domain strove to that of Occultism. One can see this when one becomes more closely acquainted with his work.

Thus it all depends on the human being rising, through the type of self-discipline indicated above, to sense-free perception. Only in this manner are the doors to Mysticism and Occultism unlocked. *One* of the paths that leads to a purification from life in the sense world is through a schooling carried out in the spirit of mathematics. And just as the mathematician first stands firmly in life, when because of his training he is able to build bridges and tunnels, that is , he is able to quantitatively master reality, so they too are only able to rule and understand the Qualitative, who have first comprehended it in the ethereal heights of sense-free perception. This is the Occultist. And just as the mathematician uses mathematical laws to mould iron structures into machines, so the Occultist shapes the life and soul in the world, using the laws of these realms which he has grasped in a mathematical way. The mathematician returns to the world with mathematical laws; the Occultist no less with his laws. And just as little as the non-mathematician can understand how a mathematician works on a machine, so just as little can the non-Occultist understand the plans by which the Occultist works upon the qualitative forms of life and soul.

*Notes* :

1. *Mathematics and Occultism* : This is a translation of 'Mathematik und Okkultismus'. The essay and the notes are from the volume 'Philosophie und Anthroposophie' -GA [Complete Edition] Bibl. No. 35, Dornach 1984, pp.7-18. It is a write-up of an address that Rudolf Steiner delivered in Amsterdam at the Congress of the Federation of European Sections of the Theosophical Society on the 24th June 1904. See Rudolf Steiner's report, 'Der

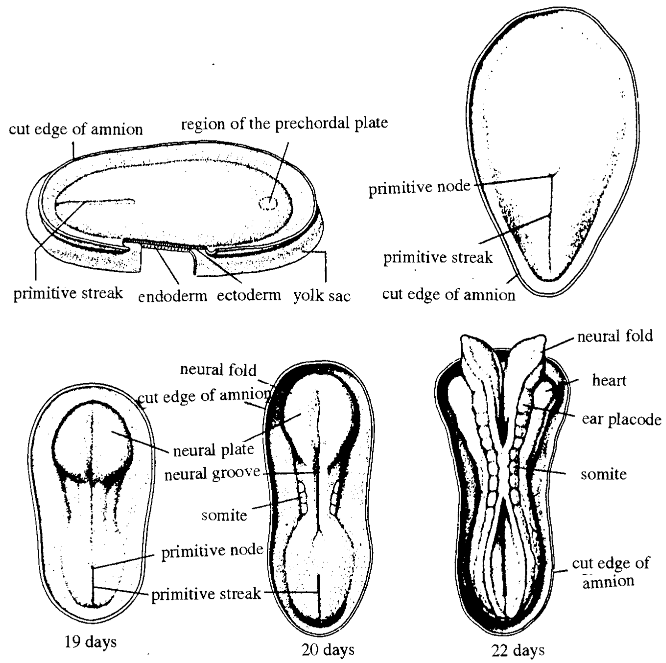
to form-less thinking. The thought of a triangle, a circle and so on, always has a form even though this form is not a directly material one. Only when we are able to pass from that which exists in a finite form, to that which does not have form, but contains the possibility of form-production within it, are we able to grasp what the Arupa-Realm is in contrast to that of the Rupa.[7] At the lowest, most elementary level we have an Arupa-Reality before us in the differential. If we calculate using the differential, we always stand at the point where the Arupean gives birth to the Rupean. With the infinitesimal calculus, therefore, we can learn to comprehend the Arupean and understand what sort of relationship it has to the Rupean. One has to only carry out in full consciousness the integration of a differential equation to feel something of the source of power that exists at the boundary between the Arupean and the Rupean. However, one has still only grasped in the most elementary way what the advanced Occultist is able to perceive in higher realms of being. Nevertheless, one has a means for beholding, and at least a *hint* of those things of which the person who remains fixed to the sense world cannot have any idea. For a person confined only to the outer senses, the words of the Occultist must at first seem to be devoid of all content and meaning.

A knowledge attained in those regions where the crutches of sense perception are lacking, can be most readily understood where the freeing from such a perception is most easily gained. This is the case with mathematics. Mathematics is therefore the most easily surmounted preparatory training for *the* Occultist who wishes to rise to bright and radiant clarity in the higher worlds, and not to a dim sentient form of ecstasy or to dreamy premonitions. The Occultist and Mystic live in the same light-filled clarity within the supersensible, as the Elementary-Geometer does in working with the laws of triangles and circles. For true Mysticism lives in the light not in darkness.

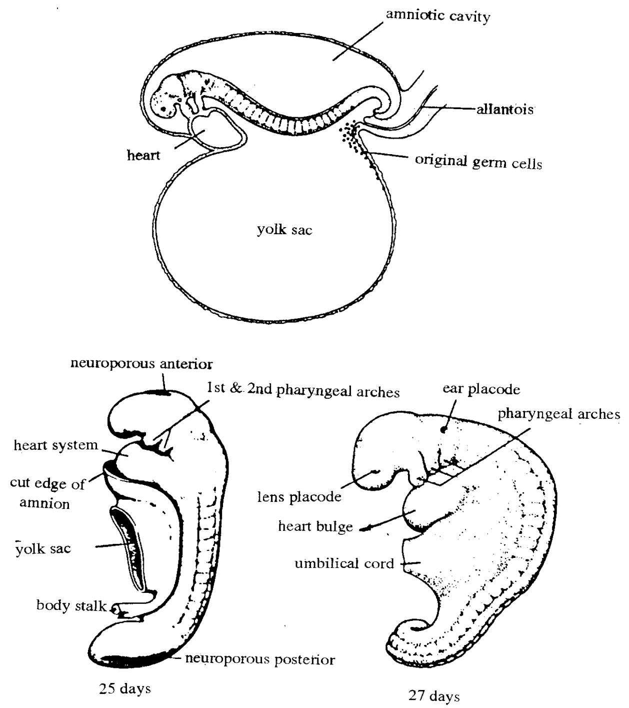
It can also be easily misunderstood when the Occultist who is speaking out of a conviction that can be called Platonic, demands research carried out in the manner of mathematics. One could assume that mathematics is being overrated. This is not the case. However, they are guilty of such an overestimation who only allow that to count as exact knowledge to the extent to which the realm of mathematics itself extends. There are natural-scientific researchers of the present time who reject every statement in the fullest sense as unscientific which cannot be expressed in numbers or figures. For them, where mathematics ends, vague belief begins; and therefore every claim to objective knowledge must cease. It is precisely those who object to this overestimation of mathematics who are able to become the true judges of that genuine crystal-clear type of research, which also proceeds in the *spirit* of mathematics, even in those regions where mathematics itself ends. For mathematics has in its most immediate sense only to do with the Quantitative. Its realm ends where the Qualitative begins.

Thus it is also a question of researching in the strictest sense within the field of the Qualitative. In this sense *Goethe* objected particularly strongly against an overestimation of mathematics.[8] He did not wish to see the Qualitative confined to a purely mathematical kind of treatment. Yet, he wished to think everywhere in the



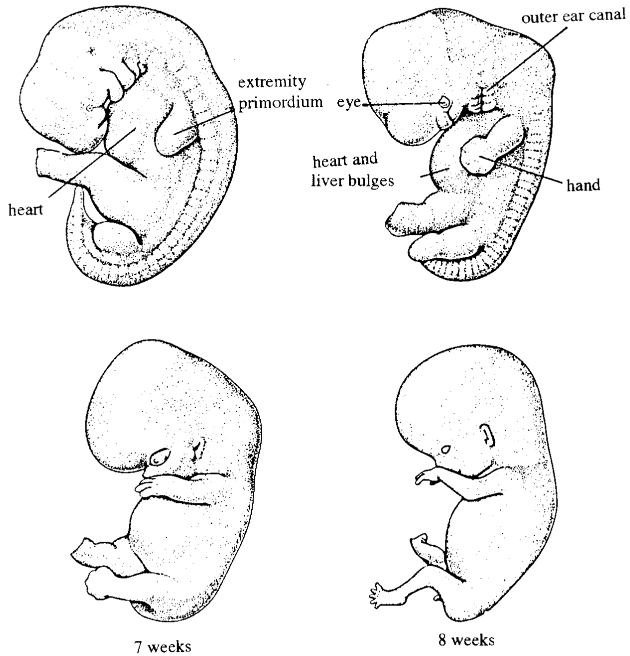



15 Top Left: Embryo slice seen sideways obliquely from above at about the 14th day. Top Right: Embryo slice seen from the back side with the head pole above, on the 18th day; yolk sac and amnion have both been removed. Bottom Left to Right: Rear views of the developing body on about the 19th, 20th, and 22nd days, respectively. The inversion of the spinal column and brain can be followed. (From Langman).




16 Development of the embryo between amniotic sac (amnion) and yolk sac in the 4th week. Upper: The large, protruding heart primordium has already been pulsating since the 21st day. Lower: The embryo begins to incurve, pharyngeal arches become visible, ear and eye formation starts; 25th to 27th day. (From Langman).

But now, what are the most important steps in the actual physical development of the embryo itself? The first constituent of this development has already been mentioned: the surface of contact between the amnion and the yolk sac. This tiny embryonic disc, only about 1 mm large, forms the primitive streak and primitive node (Hensen's node) at the posterior end, and does not change appreciably between the 14th and 17th days. At about the 18th day however, powerful metamorphoses begin. In the marginal region of the embryonic disc, and likewise in the yolk sac and chorion, the first blood islands form, which quickly flow together to form the first blood vessels, and soon form a circulation system common to the embryo and placenta; already on the 21st day the heart begins to pulse, which in the beginning is nothing more than a distended blood vessel loop. Also on the 18th day, a part of the ectoderm swells into a visible nerve structure (medullar plate), which will soon close into a tubular form, develop into the brain and spinal cord, and thus will form the basic axial body shape. Then the joint with the yolk sac narrows and becomes the site of the inner respiratory tract in the intraembryonic endoderm. In this way, the most important inner organs of the tiny growing body are formed during the second half of the third week, that is, just at the time when the mother gains the certainty of carrying a child, five weeks after her last menstrual bleeding (fertilization




17 Top Left: Even after the fifth week (35 days), the embryo is still only 7 mm long. Top Right: On about the 42nd day (13 mm long), after the heart has enlarged, the brain and liver begin to grow rapidly; arms and legs separate from the large hand and foot primordia. Bottom Left: The seven week old embryo (18 mm) becomes a foetus; the eyes move forward, the ear muscles form, the first bone deposits begin. Bottom Right: As the head becomes erect, the brain primordium becomes jammed together, the skeleton formation accelerates, the genitals become visible (30 mm crown-rump length). (From Langman).



enlivened. If we comprehend space with the outer senses, its points, its infinitely small parts are dead; but if we comprehend the points as differential quantities, there streams inner life into the dead co-existing parts. Extension itself becomes the *product* of the extensionless. Through the infinitesimal calculus life came into our knowledge of nature. The sensible is led back to the point of the supersensible.

The consequences of all that we have mentioned here are not to be grasped through the customary philosophical speculations on the nature of differential quantities, but far rather by realising through *self-knowledge* how to proceed in one's spiritual activity, when with the infinitesimal calculus, one attains the finite from out of the infinitely small. One continually stands before the moment of the *coming into being* of something sensible out of that which is no longer sensible. It is therefore entirely comprehensible that this spiritual life in supersensible, mathematical relationships of magnitude, has become a powerful educational tool for the mathematician of more recent times. We are indebted to what spirits such as *Gauss*, *Riemann* and more recently the German thinkers *Oskar Simony* and *Kurt Geissler*[5] among many others have accomplished in this field which lies beyond normal sense perception. One can always object in detail to these attempts: But the fact that these thinkers have extended our concept of space beyond the three-dimensional, that they calculated in conditions more general, more comprehensive than physical space- are all a result of mathematical thinking emancipated from the sense world by the infinitesimal calculus.

Very important indications for Occultism are thereby hinted at. For mathematical thinking continues to possess, even in those regions where it ventures beyond the sense-perceptible, the exactness and certainty that belong to genuine disciplined thinking. Aberrations may of course also occur in this domain, but their effects are never as disastrous as when the undisciplined thoughts of the non-mathematically trained penetrate into the supersensible. Just as little as Plato or the Gnostics viewed mathematics as anything other than an educational tool, so we too maintain nothing more with respect to the mathematics of the infinitely small. However, it is such an educational tool for the Occultist. It teaches him to bring strict mental discipline into those regions where false chains of reasoning can no longer be corrected with the help of normal sense perception. Mathematics teaches us to become free of the senses; however it does this via a very sure path, because although its truths are gained in a supersensible manner, they can always be confirmed through sensory means. Even if we state something mathematically regarding four-dimensional space, the statement must be of such a nature that if we were to leave four dimensions and specialise our result to three, our truth would still remain a special case of a general statement.

No one can become an Occultist who cannot carry out within himself the transition from sense-filled, to sense-free thinking. For this is the transition where we experience the birth of 'Higher Manas' out of 'Kama-Manas'.[6] This is what Plato required those to experience who wished to become his students. The Occultist, however, who has already passed through *this* experience, must still undergo something higher. He must also find the transition from sense-free thinking in forms, that a knowledge is attained through the mathematical formulation of the processes of nature which transcends sense perception, a knowledge which comes indeed to expression via sense perception, but is, however, comprehended *in the mind.* I have first comprehended the way a machine works when I have expressed it in mathematical formulae. To express in such formulations those processes lying spread out before the senses, is the ideal of mechanics, physics and is becoming ever more the ideal of chemistry. But only that which lives itself out in space and time, only that which has *extension* in this sense can be thus expressed mathematically. For as soon as one ascends into the higher worlds where it is not a question of extension in this sense, mathematics too has to renounce its immediate form. However one should not dispense with the *art* of perception which lies at the basis of mathematics. We have to acquire the capacity to speak about living things, the soul and so on, so freely and independently of their single observable forms, just as we speak of a circle independently of one drawn on a piece of paper.

Just as it is true that all our knowledge of nature contains only so much real knowledge as there is mathematics in it, so it is also true that in all higher spheres knowledge can only be attained when it is carried out after the model of mathematical knowledge.

Mathematical knowledge has made very significant progress in more recent times. It has undertaken within its own domain an important step into the supersensible. This has taken place in the analysis of the infinite, which we owe to *Newton* and *Leibniz*.[4] Thus we have received another mathematics in addition to the one we call Euclidean. Euclidean mathematics can only express in mathematical formulae those things which can be represented and constructed within the field of the *finite*. What I state regarding a circle, a triangle or in numerical relationships in the sense of Euclidean mathematics, can be constructed within the finite field and in a sense perceptible way. This is no longer possible with the differential calculus with which Newton and Leibniz have taught us to calculate. The differential has all the qualities that make it possible to carry out calculations with it, but it is as such not visible to sense perception. In the differential sense perception is first brought to a vanishing point, then we get the new sense-free foundation for our calculation. The sense perceptible is calculated out of that which is no longer visible to the senses. Thus the differential is that which is infinitely small in relation to the finite-sensible. The finite is mathematically led back to something wholly dissimilar from it - the *truly* infinitely small. With the infinitesimal calculus we stand at a very important boundary. We are mathematically led beyond the sense perceptible, but remain so much so within reality, that we can compute the imperceptible. And once we have calculated, the perceptible proves to be the result of our calculation out of the imperceptible. With the application of the infinitesimal calculus in mechanics and physics to the processes of nature, we are really carrying out nothing more than the calculation of the sensible out of the supersensible. We comprehend the sensible out of its supersensible beginning or origin. For the outer senses the differential is a point or a zero. For spiritual comprehension, however, the point becomes alive, the zero becomes a cause (Ursache). Thus, for spiritual comprehension space itself becomes



can take place at nearly any time during the monthly cycle, but usually two weeks after the last menstruation).

A new important stage of development is the transition from embryo to foetus. This happens at the end of the seventh week. The embryonic period is the period of organogenesis. Further enlargement and development of the organs occurs in the foetal period. A series of significant metamorphoses at this developmental juncture of the now about 1.8 cm long embryo can be delineated. The most striking is that the first solid substances become stored as bone material in the formerly soft, transparent living body: first in the clavicle and upper jawbone, and from there, streaming in toward the torso and head. Simultaneously, the first dark space is formed: the pigment layer of the eye has been storing melanin since the fifth week, a black pigment which now makes the eyeball into a dark *camera obscura*. Previously, the eyes were undirected, simply pointing out from either side, but now they move forwards to the position where they will point in future, where it becomes possible for these organs to focus in physical space. In the growing movements, the hands make their first grasping gestures and can touch each other for the first time. During the phase of inner organogenesis, the body remained tightly curled together, so that head and rear end were in contact, but now it unfurls and directs itself toward the touchable and visible world of the senses.

At this very moment, the developing human body is first confronted with the one-sidedness of its particular gender. This was determined by the chromosomes right from the moment of conception, but the remarkable thing is that at first, until into the seventh week, the physical development proceeds completely independently of gender. The differentiation of sexual tissue in the gonads only becomes apparent concurrently with the hardening of the body to solid mineral substance. Soon the neutral outer genital primordium polarizes into either the male or female form, so that the gender of the foetus becomes visually identifiable externally.

What is taking place here is the destiny of one of the most comprehensive processes of human evolution, one which recurrently repeats itself anew. In the book *Cosmic Memory* (GA11, 1904-1908), Rudolf Steiner describes the advances and setbacks of humanity's progenitor, readable in cosmic memory (Akasha) by spiritual researchers. The godlike spiritual powers who created the human being had originally intended it to receive a much less hardened, more malleable body. Our physical form would then always have been the direct physiognomic expression of our inner soul life. But as it turns out, this is not the case with us today. We can never completely bring our inner soul experiences to expression; rather, we hide these experiences behind the facade of the body we have been given.

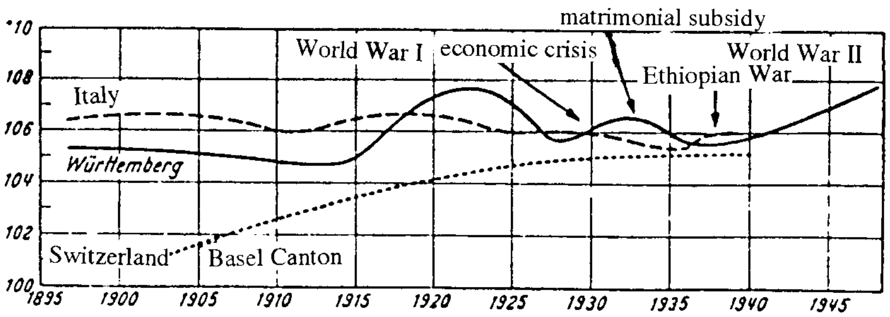
This distortion of human evolution happened in the period of old Lemuria, putting it anthroposophically, when a spiritual counter-force gave individual consciousness to humanity too early. Before the human being could act out of its own 'I', it gained the possibility of mirroring itself and of being mirrored. All discrepancies between our mental pictures and our real deeds originate from this interference by this immature spiritual seducer, the one called Lucifer by spiritual science. *Luci-*

*fer*, the light-bearer, bestowed this very light of subjective self-consciousness upon the human being, and in so doing also the cause of all egocentricity, of all alienation from the world. The result was that even the human physical bodies became denser and harder and so served better to shield the 'I'-consciousness from the world, less capable of acting beneficially for the world. Thereby, the world became the sensory objective outer world, opposed by the 'I' as subject. "Then their eyes were opened" is the expression used in the myth of paradise, from which the human being was subsequently expelled.

With material solidification, the body also lost its formerly autonomous ability to produce offspring. Only part of what is necessary for reproduction could still be generated, and completion by the other gender became necessary. The originally androgynous, hermaphroditic human being fell into the restriction of unisexuality because the body's vigorousness became impoverished through the luciferic hardening. The human foetus of the seventh week of pregnancy bears the decisive imprint of this hardening. The cosmically-oriented being became a body restricted to the earthly.

In the Platonic dialogue about love, in *Banquet* (Symposium), Aristophanes tells of this in the mythical image of the hermaphrodite, who, cut in two by Zeus, was divided into a male and a female human. The word *sexus* means "being cut" (from *secare*, to cut), so that even today the word sexuality calls this one-sidedness by its name.

Recalling that gender is determined for every cell right from the moment of conception, the effect of this cosmic event becomes manifest. The simple Mendelian interpretation of an "accidental combination" of the egg cell with the sex-determining spermatozoa (X-sperm produce a female zygote; Y-sperm yield a male zygote), based on the distribution scheme of backcrossing, is evidently not entirely accurate. The girl : boy proportion is already at 100 : 106.3 at birth; for the foetal period it is even as skewed as 100 : 120 or up to 130. After both world wars, with their respective heavy blows to the male population, the birth rate of boys in the countries participating in the wars increased sharply, but this was not the case in neutral countries, like for ex-




18 Variations in the proportion of births of boys to 100 girls in some European countries between 1898 and 1948. (From Conrad). [Translator's note: A matrimonial subsidy is a cheap loan given by the state upon marrying, whose principal was reduced with the birth of each additional child.



the sense perceptible circle which teaches me the laws of the circle, but the ideal one, that lives in my mind and of which the sense perceptible is only an image. Every other sense perceptible image of the circle can also teach me this. That is what is so significant about the mathematical conception: that a single sense formation leads out beyond itself; and it can only be for me an image of an all-embracing spiritual fact. And thus, there also exists the possibility of bringing the spiritual in this sphere into the domain of the sense perceptible. For with mathematical forms I can become acquainted with supersensible facts through sensory means. That is what was so important for Plato. The Idea has to be perceived purely spiritually if it is to be known in its true essence. One can arrive at knowledge about these things by exercising the first steps in mathematics, and by becoming aware of what it is that one actually gains from a mathematical form. Learn with the help of mathematics to make yourself free of the senses, for only then can you hope to rise to the sense-free understanding of ideas. - That is what Plato wished to instil into his students.

The Gnostics, for example, also demanded something similar. "Gnosis is Mathesis," is what they said. They didn't mean, however, that the essence of the world is to be founded upon a mathematical view, but only that the *first step* in the spiritual education of humanity lay in this view of attaining the supersensible. A person is on the path to spiritual knowledge when he has reached the stage of thinking sense-free about other properties and qualities of the world, just as he has learnt with Mathesis to think about geometrical forms and arithmetical number-relationships. The Gnostics didn't strive after Mathesis itself, but rather for spiritual knowledge attained *after the model* of Mathesis. They saw in Mathesis a standard or a model, and because the geometrical relationships of the world are the most elementary and simple, they are therefore the easiest for a human being to attain. One should learn with elementary mathematical truths to become free of the senses, so that one can also later on become sense-free when dealing with questions of higher knowledge. - Certainly for many the dizzying heights of human knowledge are thus indicated. However, those whom we can designate as true Occultists, have always demanded of their students the courage to make these dizzying heights their own. "You may only become a student of the Mysteries, when you have learnt to think about the Being of Nature and Spiritual Existence wholly free of any sense-perception, just as a mathematician does regarding a circle and its laws." That should stand in golden letters before everyone who really seeks the truth. "You will never find a circle in the world, which doesn't confirm within the sense world that which you have learnt in sense-free mathematical perception; no sense experience can deceive you regarding your spiritual knowledge. You attain, therefore, an imperishable, an eternal knowledge, once you know yourself to be free of the senses." Mathematics is therefore thought of as an educational tool by Plato, the Gnostics and all true Occultists.

We will now consider what some outstanding personalities have had to say on the relationship between mathematics and natural science. There is only so much true science in our knowledge of nature, as there is mathematics in it - has said, for example, Kant[3] and many like him. Nothing else is thus indicated, except to point out

## Mathematics and Occultism[1]

Rudolf Steiner
Translated by David Wood

It is well known that the heading above the Platonic Academy was said to have excluded anyone ignorant of mathematics from participating in the teaching of the Master. Now, whatever one may think about the historical truth of this tradition, the fact is that it contains the correct feeling for the position which Plato[2] held mathematics to hold within the sphere of human knowledge. Through his 'Doctrine of Ideas' Plato wished to teach his students to move with their knowledge in the world of pure spiritual being. He believed that man can know nothing of the *true* world, as long as his thinking is permeated by that which the senses transmit. He demanded sense-free thinking. A person moves within the world of ideas when he has separated everything out of his thinking that sense perception can provide. Above all for Plato the question arose: How does the human being free himself from all sense perception? He regarded this as an important question for the education of the spiritual life.

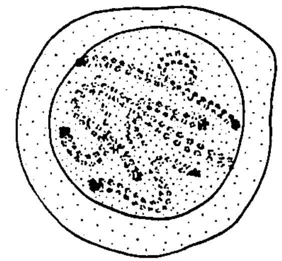
It is only with difficulty that the human being liberates himself from sense perception. Self-knowledge can teach us this. Even when the average person withdraws into himself, to let no sense impression work upon him, there are still, nevertheless, traces from the senses present. And the undeveloped person stands before a void, a completely empty consciousness, if he disregards the impressions received from the sense world. It is for this reason that certain philosophers maintain there is no such thing as sense-free thinking. For even if a person completely withdraws into the field of pure thought, he is still only occupied with the fine shadow-pictures of sense perception. But this statement is only valid for the undeveloped person. As soon as a person acquires the ability to develop within himself organs of spiritual perception, just as nature has developed within him organs of sense perception, his thinking ceases to remain empty when all sensory content is separated out. Such sense-free, and yet spiritually-filled thinking, was what Plato demanded from those who wished to understand his Doctrine of Ideas. Thus he had only demanded something, which teachers in all ages *have had* to demand from their pupils, when they have wanted to make them into true initiates of higher knowledge. If a person has not experienced in all its ramifications that which Plato demanded, he will be unable to attain any insight into the true nature of wisdom.

Plato regarded the mathematical view as an educational means for living within the sense-free world of ideas. For mathematical forms hover on the boundary between the sense world and the purely spiritual world. Think of a 'circle'. One does not thereby think of this or that sense perceptible circle that one has perhaps sketched on a piece of paper, but any and every given circle that can be drawn or met with in nature. This is the case with all mathematical forms. They relate to the sense world, but are in no way exhausted through anything found in it. They hover over innumerable and manifold sense formations. When I think mathematically I think about the things of the sense world, but at the same time I do not think *within* the sense world. It is not



ample in Switzerland.

As we know today (Langman), the physiological reproduction process itself is generally not concluded by the first sperm which reaches the egg cell. On the contrary, countless spermatozoa with protein-dissolving enzymes first locally dismantle the covering of nurse cells (corona radiata) and the already mentioned jelly membrane (zona pellucida), before fertilization can take place between the ovum and a sperm cell working in concert. The fact that cells react most sensitively to outer influences during cell division is well known to cell physiologists. In particular, during cell division exactly that part of the nuclear substance relevant to heredity is most sensitive (for example to mutagenic influences such as X-rays). It is most remarkable that with the beginning of foetal life in the third month, the future ova in the ovary of a female foetus enter the first phase, the prophase, of meiotic division and will remain in this state for up to 50 years—except for those oocytes which complete their first meiotic division during the successive menstrual cycles commencing at puberty. After an egg is released during ovulation, the second meiotic division begins and then halts in the middle of cell division, in the metaphase, until fertilization succeeds. Only then can the second meiotic division be brought to an end and the final ootid be formed, and otherwise the ovum atrophies within twenty-four hours. So while waiting to complete the first meiotic division as well as while waiting for fertilization, the corresponding ovoid halts in its most sensitive position, namely of incomplete meiosis: at these junctures, an ovum is maximally open to all influences which may work upon it.



19 Primary oocyte in the prophase of the first meiotic division with visibly reduplicated (doubled) chromosome strands. It remains in this state from the third foetal month for many years or decades, until eventual egg maturation during one of the menstrual cycles. (From Langman).

Based on 33 000 births registered in Freiburg in the years between 1925 and 1938, Walther Bühler was able to deduce that female birth-rates exceeded the normal two to three days after the *full* moon and that male birth-rates surged two to three days after the *new* moon. This investigation was inspired by a description by Rudolf Steiner, who said that the choice of gender is made shortly before incarnation, during the transition through the spiritual moon-sphere (GA 218). Like with an organism, the moon is not merely matter in empty space. The moon’s varying position relative to the earth and sun in its phases of change is simultaneously a variation in the supersensible constellation of the nearer cosmic perimeter of the earth. Let us try to find some evidence of this.

Wherever something outwardly visible dwindles, super-sensible room is provided for spiritual happenings. Where the outer, illuminated surface of the countenance of an acquaintance stops, in the black pupils of his or her eyes, we experience the presence of his or her gaze. Rudolf Steiner described something of this in a broader sense for the cults of the ancient priests of the high Stone Age culture (GA 227, 228). From out of the space flooded by sunlight, the menhir cut a shadow space in which the light of the spirit became visible to the honest seeker. Furthermore, the cults of ice age peoples took place in the darkrooms of deep caves, where far removed from the sunlit world, magical pictures of the hunt lit up the mind and were painted onto the walls of the remotest rooms.—Even today in Africa an honoured person is referred to by: "Nandisipoh truly casts a shadow." (van der Post). And Adalbert von Chamisso fictionally characterized shadow space in his *Peter Schlemihl*, who, in selling his shadow, unwittingly also sold his higher being.

The spiritual human being, drawing itself together from the cosmos, metaphysically wanders through the planetary spheres toward the earth, in order to pass through the threshold of the moon sphere into earthly space, into the sublunary sphere. If a female incarnation is desired, full moon is the moment which is chosen for it—when the shadow space of the moon is opened out toward the cosmos, pointing away from the earth. In comparison, the female constitution remains more cosmic than the male. The male form of incarnation is determined when the shadow space of the moon points toward the earth, at new moon. And thereby the sensitivity of the ovum is tinged by the selective acceptance of the gender-determining spermatozoa. The wide temporal range of possible moments of conception over the duration of the female monthly cycle, as well as the deviation of the actual date of birth (within two weeks before or after the date of birth calculated from the last menstruation), makes it very difficult to discern the correspondence of gender distribution with the synodic phases of the moon. But the evidence is already overwhelming enough.

Finally, let us come back to the special spiritual significance of the embryonic and foetal membranes. Here it is possible to raise the objection that even mammals grow and develop in such membranous organs and yet they do not become humans. Pursuing the facts, it becomes clear that a stepwise accumulation of membranous organs is observable, which increases with the evolutionary stage of the chordates. Fish and amphibians, as lower chordates, have a yolk sac and with this, through an undeniable central nervous system, also an undeniable inner life (on the other hand, all invertebrates, all non-chordates, always have only diffuse ring-shaped or double-stranded nerve networks). The higher vertebrates, the reptiles and birds, gain the amnion and allantois. The amnion (amniotic sac) replaces the external water pond of the fish and frogs, the allantois (the urinal membrane) mediates fluid and gas exchange via the existing egg shell. With the mammals and the human being, the chorion (or the placenta, the villous remainder of the chorion) appears as the final new membranous organ. In this way it becomes clear that the animal kingdom visibly expresses increasing stages of evolution right in the most spiritually permeated organs—in the extraembryonic membranes. Now that we have established that the consummation of all this



4 For a treatment of the concept of 'counterspace', see for instance: G. Adams and O. Whicher, The plant between sun and earth, and the science of physical and etherial spaces, Shambala publications inc., Boulder (Colorado), 1982.
5 For the concept of 'reciprocating' or 'polar-reciprocal inversion' I refer to the basic literature on projective geometry.
6 In this equation, $o_2a$ is the point of intersection of $o_2$ and $a$, $o_2b$ that of $o_2$ and $b$, $o_1c$ that of $o_1$ and $c$ and $o_1d$ that of $o_1$ and $d$. Since $a$ and $b$ are parallel, the distance between $a$ and $b$ equals the distance between $o_2a$ and $o_2b$. Similarly, the distance between $c$ and $d$ equals the distance between $o_1c$ and $o_1d$.
7 Translation D. Sprangers.

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*With this article the author submitted another entitled 'Elements of a differential and integral calculus of counterspace'. This will be published in the next issue. However, in the interim, the author is willing to supply a copy to anyone who is seriously interested.*

view, is shown, among other places, by the following sentences from the book ‘Geheimwissenschaft im Umriß’ (p. 112 in the edition mentioned). They describe the sight of a crystal, as it presents itself to a person who can observe it spiritually: ‘Nur verhält sich dasjenige, was sich da offenbart wie ein Gegensatz dessen, was in der Sinnenwelt auftritt. Der Raum welcher in der letzteren Welt von der Gesteinsmasse ausgefüllt ist, erscheint für den geistigen Blick wie eine Art Hohlraum; aber rings um diesen Hohlraum wird die Kraft gesehen, welche die Form des Steines bildet.’ (‘Only that which reveals itself here, is like a contrast to that which occurs in the sense world. The space which, in the latter world, is filled up with the stone mass, appears to the spiritual view as a kind of hollow space; but round about this hollow space the power is seen, which forms the shape of the stone.’[7]) If one wishes to include this ‘outer space’, in which the formative forces reveal themselves and which is also indicated as ‘counterspace’ or ‘negative space’, into the consideration, then the attention must also be directed to it when dealing with the warmth processes. In the teaching of physics I have done this by directing the attention to the totality of all perpendiculars which we can drop on the surface of for instance a cubical crystal. In this exercise, a corner was considered as one eighth of a small ball with a radius approaching zero, and an edge as one fourth of a small cylinder with a radius also approaching zero. From the picture of the totality of all these perpendiculars, it turns out that the ones ending at the corners take up such a large portion of the outer space that the portion of all the other parts sinks into insignificance beside it (mathematically: it is infinite times as large). This picture creates the impression that the interaction between the outer and the inner space takes place predominantly at the corners. When for instance, upon heating, the outerspatial formative forces pull the cube to the periphery, the corners make up the most important points of application for these forces. Thus, if their contribution is ignored, one ‘streicht damit weg das Allerwichtigste, worauf es ankommt’.

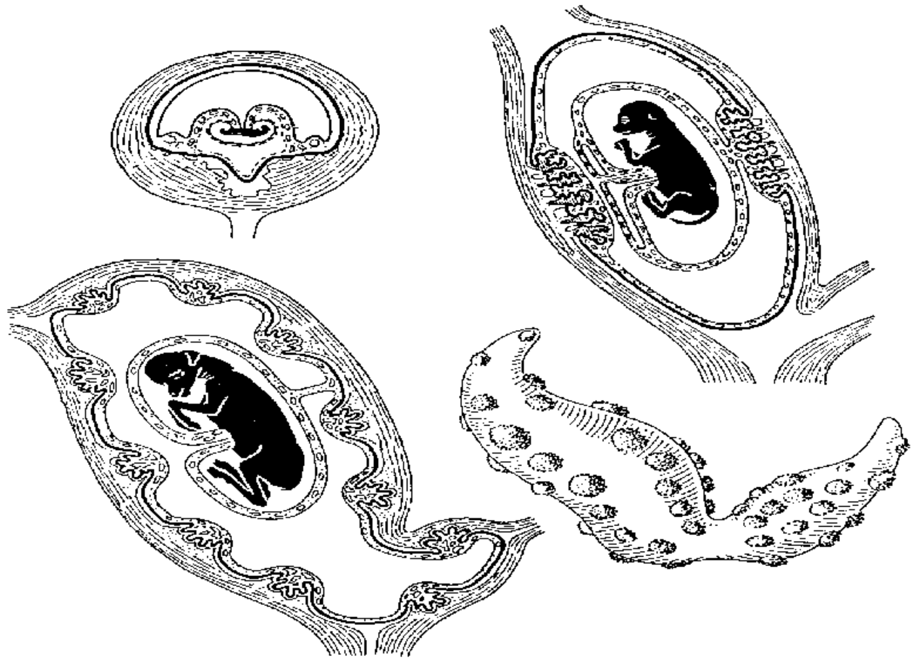
This explanation may make it understandable that learning about the article of P.P. Veugelers filled me with great joy. I have the impression that by treating counterspace in such a ‘professional’ way, he opens up the possibility of a very exact approach to it. And the importance of this cannot be overestimated. For the study of the lectures of Rudolf Steiner with a natural-scientific character has given me and many others the impression, that the insights in this field can only liberate themselves from the bane of mechanistic materialism, if we manage to balance in our thinking the negative-material counterspace with the positive-material space.

March 1997, Dutch Original, October 1996

1 R. Steiner, Geisteswissenschafliche Impulse zur Entwickelung der Physik, Zweiter naturwissenschaftlicher Kurs, Die Wärme auf der Grenze positiver und negativer Materialität, GA 321, Rudolf Steiner Verlag, Dornach (Switzerland), 1982.
2 Translation D. Sprangers.
3 G. Adams, Wärmeausdehnung im Aetherraum, Mathematisch Physikalische Korrespondenz 2, 1954.



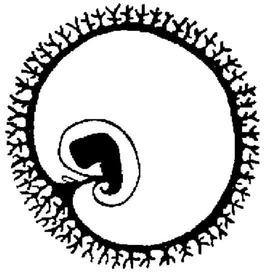
radical, transformative development is the very being of the human, we may here remark that the human condition did not commence with us, so to speak "beyond the chimpanzee;" instead, it was already delineated in the various stages of animal evolution. In this sense, the higher animals are not mere "beasts," but rather, seen evolutionarily, also human. They appear to be more or less distantly related to modern human beings because they ceased evolving earlier and became adapted to specific niches. With increasing proximity to the human being, the extraembryonic membranes become increasingly refined, demonstrating the direct participation of the animal kingdom in our common human evolution.



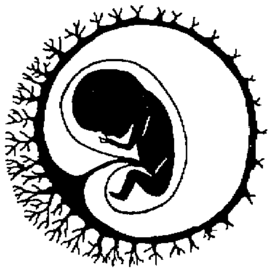
20 Extraembryonic membranes in the uterus of a mouse (above left), dog (above right), cow (below left), the latter depicted with its closed placenta (below right), revealing its many individual patches of chorionic villi. Observe the extensive placenta of the ungulates, the belt-shaped placenta of the carnivores, and the cuneate, centralized placenta of the rodents. (From Portman).

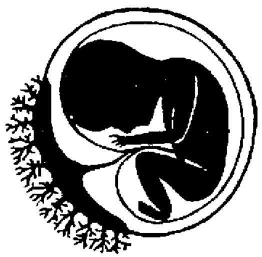
Looking at the intricate placental construction of the different mammals, some additional impressive facts are revealed. Noticeably all smaller mammals, especially the rodents, develop only a very restricted placental organ, the so-called placenta arconus, which follows the central motif of the development of their physical bodies. But nothing but small, weak, nervous little bodies are the result! In contrast, many ungulates form placentas with villi uniformly distributed over the entire surface, which are maintained for long periods of time, sometimes even for the whole gestation period.

In this way, the high spirituality of the ungulate group-souls is expressed, powerfully connected to the cosmic periphery. Strong, lively constitutions are formed in a large "amniotic sac," so that later, in contrast to the nearly mineralized faeces of mice, they can pass the benefits of their rich dung on to the plant kingdom all their lives long. Then the placenta of the carnivore: although it encloses the animal young, it does so only partially, in the form of a zonary, belt-like placenta, the one typical for them. Finally, the apes construct two placentas right from the beginning, a primary and an auxiliary placenta (for more about mammalian placentation, see Schad, 1971).



21 Formative transformation of human placentation. From the omnidirectional, *diffuse* placenta, a bounded discoidal placenta (placenta discoidalis) gradually emerges. Above: 1½ months, Middle: end of the second month, Below: end of the third month. For comparison, the relative blastular size has been maintained. Similar proportions hold for anthropoids, but not for other apes.





With this in mind, if we now turn to the placentation of the human being, we can observe its formation and how it constitutes a human being. Initially the placenta is distributed uniformly around the entire surface of the chorion sphere: the direct expression of an all-encompassing spiritual origin emanating from the cosmos. However after the threshold of the seventh week, the placenta gradually restricts itself (with all its villi) to the location in the uterine lining where the foetus was originally attached. At this time, the placenta also becomes active in the formation of the physical, earthly human being, as becomes apparent in the flat, centralized discoidal placenta, which ends up becoming about 15-25 cm in diameter. The higher human being inclines to the tasks of the forthcoming earthly life, just as the developing earthly body keeps a connection to its sphere via the umbilical cord. So, unlike any animal, the incarnating human being, the human being becoming flesh, preserves the full, enduring tension between being able to grasp and mould the entire earth and having to search in his or her supersensible being for a responsible position and task in the context of the whole world. More than at any time in later life, human prenatal existence is the invisible, visible expression of this tension. These new epistemological pictures of the fundamental laws of human embryological development, to the extent that they have been given to us by modern natural and spiritual science, are necessary today to attain to a new responsible relationship to unborn life.



Rudolf Steiner's consideration gave the insight for the essence of the warmth processes, but that only the starting point contained an – essentially unimportant – error. With an open mind, I could address myself to the fact that this consideration shows very clearly that the attention is directed to a triplicity here: the positive (material) space, the counterspace (formative forces) and the essentially unspatial. This triplicity is fundamental to the complete series of lectures.

On closer inspection of the correspondence between the formula

$V_t = V_0\left(1+3\alpha t+3\alpha^2 t^2+\alpha^3 t^3\right)$ and the spatial change of the inner space of a cube

which is heated from 0°C to $t$°C, it turns out that each of the three terms of the volume increase $\Delta V_t = 3V_0\alpha t+3V_0\alpha^2 t^2+V_0\alpha^3 t^3$ is related to particular parts of the spatial change (see figure).

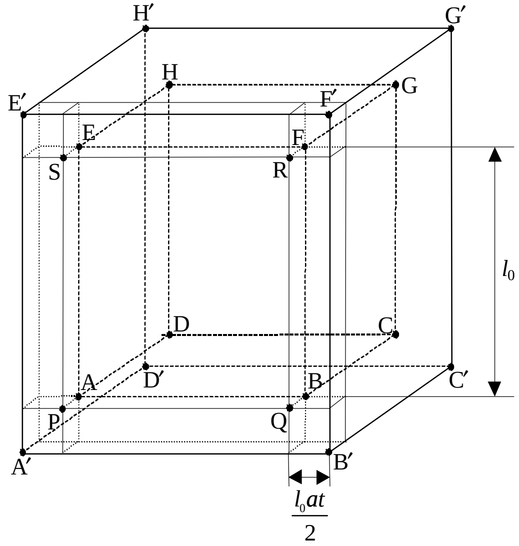
Cube at 0°C: ABCD.EFGH; edge $l_0$.
Cube at $t$°C: A´B´C´D´.E´F´G´H´; edge $l_t = l_0+l_0\alpha t$.
$DV_1 = 3V_0\alpha t$: first part of the volume increase: rectangular blocks such as ABFE.PQRS.
Volume per block:

$l_0^2\,\frac{l_0\alpha t}{2} = \frac{1}{2}l_0^3\alpha t$ . 6 blocks.

$DV_2 = 3V_0\alpha^2t^2$: second part of the volume increase: rectangular beams such as those between the pairs of edges AE.A´E´, AB.A´B´, BF.B´F´, FE.F´E´.
Volume per beam:

$l_0\left(\frac{l_0\alpha t}{2}\right)^2 = \frac{1}{4}l_0^3\alpha^2t^2$ .

12 beams.




$DV_3 = V_0\alpha^3t^3$: third part of the volume increase: small cubes at corners such as A, B, F, E. Volume per small cube:

$\left(\frac{l_0\alpha t}{2}\right)^3 = \frac{1}{8}l_0^3\alpha^3t^3$ . 8 small cubes.

However, this describes only the material inner space of the cube. The complete outer space has been left out of consideration. This contains the whole space except for the inner space. How important this outer space is from a spiritual-scientific point of

$t$°C is: $l_t = l_0 + \Delta l_t = l_0 + l_0 a t = l_0(1 + a t)$ . The contemplation of the so-called ‘dimension’ of the term $at$ produces the following result: $a$ is the ratio of two lengths (which is dimensionless) per degree Celsius and therefore has the dimension

$\frac{1}{t} = t^{-1}$ . Thus, the term $at$ is dimensionless, as is the other term 1 in the formula.

This consideration of dimensions does not occur in the lectures mentioned, but the Goetheanistic physics teacher can, no more than any physics teacher, ignore it.

Moving on to the volume increase of a cube upon being heated from 0°C to $t$°C, the length of an edge grows from $l_0$ to $l_t$ and the volume from $V_0 = l_0^3$ to $V_t = l_t^3$ , from which it follows that $V_t = l_0^3(1 + at)^3 = V_0\left(1 + 3at + 3a^2t^2 + a^3t^3\right)$ .

The same consideration of dimensions again shows that the terms $3a^2t^2$ and $a^3t^3$, as well as the term $3at$, are dimensionless.

This is where my struggle started with Rudolf Steiner's indication which is related to the second and – especially – the third power of the temperature: ‘Es ist außerordentlich wichtig, daß gerade festgehalten werde an diesen Umständen, daß wir hier bekommen die dritte Potenz der Temperatur.’ (‘It is exceptionally important that one adheres to these circumstances, that we obtain here the third power of the temperature.’[7]) When Rudolf Steiner expresses his serious objection to the usual practice of ignoring the terms $3a^2t^2$ and $a^3t^3$ because of their low numeric values, he states: ‘Nun, meine liebe Freunde, damit streicht man weg das Allerwichtigste, worauf es ankommt, wenn man nun wirklich sachgemäß Wärmelehre treiben will.’ (‘Now, my dear friends, in doing so one crosses out the most important thing, which matters, if one really wants to cultivate the theory of heat in keeping with the facts.’[7]) In the third lecture he continues this consideration. The struggle I referred to was based on a discrepancy which presented itself to me: on the one hand I had to consider the terms $3at$, $3a^2t^2$ and $a^3t^3$ just like the number 1, as being dimensionless, on the other hand Rudolf Steiner – a spiritual ‘titan’ – treats them as if they have the dimension of a temperature. This struggle lasted many months, because I could assume that his consideration was based on supersensible observations and thereby opened the way to insight in the essence of warmth. An opening in this rather oppressing situation came when I read the following sentence from the book ‘Die Geheimwissenschaft im Umriß’ (on p. 143 in the 26th edition): ‘Es kann sogar vorkommen, daß ein Forscher, der auf übersinnlichen Gebieten wahrzunehmen vermag, sich Irrtümern in der logischen Darstellung hingibt, und daß einen solchen dann jemand verbessern kann, der gar nicht übersinnlich wahrnimmt, wohl aber die Fähigkeit eines gesunden Denkers hat.’ (‘It may even happen that a researcher who is able to perceive in supersensible areas, gives way to errors in his logical presentation, and that somebody like him can then be corrected by a person who does not perceive in the supersensible at all, but does have the capacity of a sound thinker.’[7]). This made me consider the possibility that



*References to Steiner’s Collected Works (GA numbers):*

GA 35: *Philosophy and Anthroposophy, Mercury Press, Spring Valley, NY, 1988*

GA 205: *Human Development, World Soul and World Spirit, Part One* (thirteen lectures; Stuttgart: 16 June; Bern: 28 June; Dornach: 24 June to 17 July, 1921) [Menschenwerden, Weltenseele und Weltengeist - Erster Teil].

GA 218: *Spiritual Connections in the Formation of the Human Organism* (sixteen lectures: different locations, 14 October to 9 December 1922) [Geistige Zusammenhänge in der Gestaltung des menschlichen Organismus].

GA 226: *Man’s Being, his Destiny and World Evolution, Anthroposophic Press.* (seven lectures; Kristiania (Oslo): 16-21 May 1923) [Menschenwesen, Menschenschicksal und Welt-Entwicklung].

GA 227: *The Evolution of Consciousness, Rudolf Steiner Press* (thirteen lectures: Penmaenmawr, 16-31 August, 1923) [Initiations-Erkenntnis].

GA 228: *Initiation Science and Star Knowledge* (eight lectures: different locations, 27 July to 16 September, 1923) [Initiationswissenschaft und Sternenerkenntnis]. 5 lectures in *Man as a Picture of the Living Spirit (Rudolf Steiner Press) and Man in the Past, Present and the Future and the Sun Initiation of the Druid Priest and His Moon Science (Rudolf Steiner Press)*

GA 316: *Meditative Observations and Directions for Deepening the Art of Healing* (eight lectures; Dornach: 2-9 January 1924) [Meditative Betrachtungen und Anleitungen zur Vertiefung der Heilkunst]. *Available in English as study material from the Anthroposophical Medical Association of Great Britain under the title ‘Lectures to Young Doctors’.*

*Schad W.* (1971) Säugetiere und Mensch, Zur Gestaltbiologie vom Gesichtspunkt der Dreigliederung. Verlag Freies Geistesleben, Stuttgart. *Man and Mammals: Towards a Biology of Form.* Garden City, New York: Waldorf Press, 1977.

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# On the lemniscatory motion of sun and earth

by Louis Locher-Ernst

*First appearing more than half a century ago in Mathematisch-Astronomische Blätter no. 4, Nov. 1942, published by the Mathematisch-Astronomischen Sektion, Freien Hochschule für Geisteswissenschaft, Goetheanum, Dornach, this article was written as a kind of conclusion to a series of lectures by Rudolf Steiner reproduced in the same journal.* Translated by Paul Courtney.

Astronomers and mathematicians could base their life's work on the twelve lectures [Rudolf Steiner, *Entsprechungen zwischen Mikrokosmos und Makrokosmos. Der Mensch—eine Hieroglyphe des Weltenalls*, Dornach, April to May 1920] in Nos. 2 to 4 of *Mathematisch-Astronomische Blätter* and grow in wisdom as a result, the lectures are so full of new ideas, observations and references to new research areas. The cycle Rudolf Steiner gave nine months later in Stuttgart chiefly for the Waldorf school teachers of that time, *Die Verhältnis der verschiedenen naturwissenschaftlichen Gebiete zur Astronomie* (18 lectures, 1 to 18 Jan. 1921, GA 323), contains amplifications, additions and remarks from completely different aspects, particularly in relation to the living geometry of the human form. The lectures of April and May 1920 form a kind of introduction to this course[1]. If possible — this will depend on the interest shown — further remarks and elaborations on the mathematical and astronomical content of Rudolf Steiner's lectures should appear in the next numbers of *Mathematisch-Astronomische Blätter*.

We cannot go into all the naturally arising questions in this brief conclusion, still less is it possible to elaborate on individual references in this number. So we shall select just one question — a pre-question as it were — since the astronomer, as well as the layperson with some knowledge of astronomy, will ask one question above all others: Is it possible to find any agreement between e.g. lemniscatory motion and physical observation? It has to be said that to begin with there is a difficulty. Until this difficulty is overcome, a severe restraint will weigh on anyone who wants to try the new path of research opened up here. Viewed from without, the task which the mathematician must face here is not particularly rewarding however. First, most academics today are not inclined to extend their studies to include the human being in the way shown here. On the other hand there are unfortunately many laypeople who believe they can do without an exact study of today's astronomy and are content to swim about in new ideas they themselves have never discovered. Rudolf Steiner himself remarked in the clearest terms in the course mentioned above that these things could be stated in precise mathematical terms, the purpose being to express both outer observations *and* perception of the connection between human being and cosmos in a vivid way. To be sure it is necessary to work with various different geometries at the same time, geometries which go far beyond what most of us know from school. The mathematics of the last decades is capable of doing that, though so far no one has taken this up seriously.



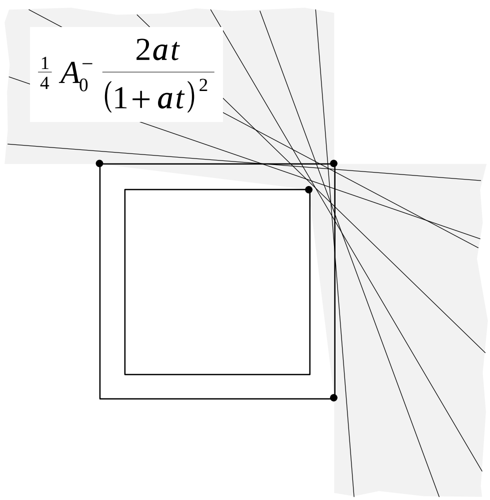



**Figure 4.2**

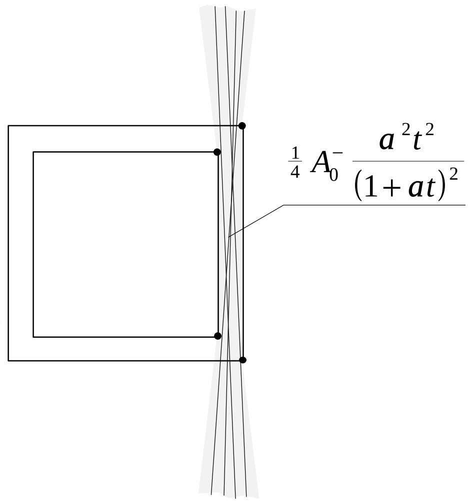



**Figure 4.3**

It turns out that also in the equation for thermal expansion in counterspace, which is derived in this article, ignoring the quadratic term is all right from an arithmetical point of view. I consider it likely, however, that Steiner, with the quoted remark, did not aim at an arithmetical aspect but at a phenomenological aspect: namely that by ignoring the quadratic term in the equation for the Euclidean space, *the contribution of the corners to the total phenomenon is implicitly ignored*. And this is exactly what is *not* justified if one includes the counterspace in the argument. For, as I have proved, *in counterspace the corners are just what contributes most* to the phenomenon.

## ***Postscript*** (By R. van Romunde.)

I was very pleased to learn the contents of this article, because it holds a strictly mathematical confirmation of a line of thought which developed in my mind as the result of a struggle to gain an insight into the deeper background of the contemplation which Rudolf Steiner gave in the second and third lectures of the 'Wärme-lehre', held on 2 and 3 March 1920 and quoted by the author of this article. In these lectures, Rudolf Steiner first treats the linear expansion (e.g. of a rod) and then that of a volume (e.g. of a cube). In doing so, he follows the current mathematical treatment, which is based on the linear coefficient of expansion a. If a rod at 0°C has a length $l_0$ and its length increase upon being heated to $t$° C is $Dl_t$, then this coefficient

$a = \frac{\Delta l_t}{l_0 \cdot t}$ ; in words: a is the ratio between the length increase upon heating and the original length per degree Celsius. From this, it follows that the length $l_t$ of the rod at

what we can say arithmetically about the magnitude of these terms.
Thus, it appears that both in equation (3.21) and in equation (3.23) the terms of higher order may be ignored. An important difference, however, is that in equation (3.21) the remaining term $V_0^+ \cdot 3at$ is the volume between the *sides* of the cubes, whereas in equation (3.23) the remaining term $V_0^- \dfrac{3at}{(1+at)^3}$ is the counterspatial volume between the *corners* of the cubes. Unlike in the Euclidean space, therefore, the corners contribute most to the volume change in counterspace.

### *4 Conclusion*

In part 3, equations have been derived for the expansion of a square in the Euclidean space and in counterspace. For the expansion in the Euclidean space, we have:

$$A^+ = A_0^+\left(1+2at+a^2t^2\right) \qquad \textbf{(4.1)}$$

In this equation, the term $A_0^+ \cdot 2at$ corresponds to the expansion along the sides of the square and the term $A_0^+ \cdot a^2t^2$ to the expansion at the corners (see figure 4.1).

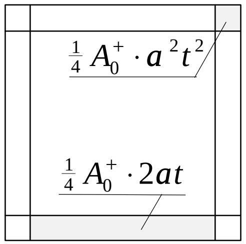



**Figure 4.1**

Thus, ignoring the quadratic term, which is usually done, amounts to ignoring the contribution of the corners to the total expansion. This is justified if one considers only the Euclidean space.

For the expansion of a square in counterspace we have:

$$A^- = A_0^-\left(1-\frac{2at}{(1+at)^2}-\frac{a^2t^2}{(1+at)^2}\right) \qquad \textbf{(4.2)}$$

In this equation, the linear term $A_0^- \dfrac{2at}{(1+at)^2}$ corresponds to the expansion at the corners of the square (figure 4.2) and the quadratic term $A_0^- \dfrac{a^2t^2}{(1+at)^2}$ to the expansion along the sides (figure 4.3).

Since the latter term is an order of magnitude smaller than the former, the quadratic term in equation (4.2), too, may be ignored. In this case, however, one does not ignore the expansion at the *corners* but that along the *sides* of the square.

For the two-dimensional situation, the remark of R. Steiner quoted in the introduction means that in the equation for thermal expansion in the Euclidean space, it is not right to ignore the quadratic term. However, it is evident that from an arithmetical point of view, there is nothing wrong with doing so. Steiner must therefore have meant something else.



We return to our modest question: whether it is possible, in conformity with observation, to use lemniscatory orbits as a basis for the motion of Sun and Earth. We shall show briefly that the answer is Yes. In the first section the principle is described in a readily understandable if somewhat simplified form. The second section gives a more detailed formulation.

I. Picture a point running round a circle *ABCDA* with constant speed (Figure 1). Imagine in each position a vertical chord drawn through the point. Now look at the point of intersection of this chord with the horizontal diameter. As the point runs uniformly round the circle from *A* to *B*, this point of intersection of the chord moves from *A* to the centre of the circle, at first slowly, then faster. At the circle's centre itself it has the same velocity which the point *on* the circle has at *B*. Thus, corresponding to the rotation round the whole circle, there is a movement to and fro on the diameter *AC*, a movement which obeys a law. This is called *harmonic* oscillation.

Picture now a point *E* executing this harmonic motion on the horizontal diameter of

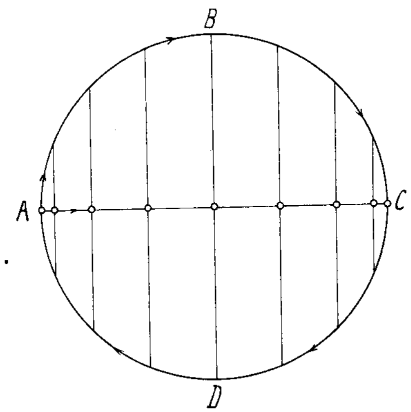



Fig. *1*

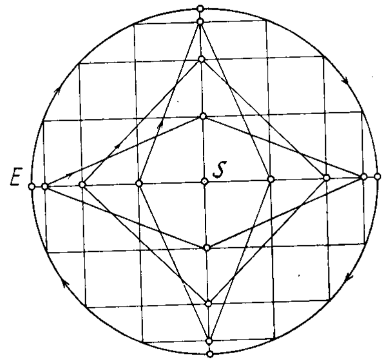



Fig. *2*

Figure 2. Suppose at the same time another point *S* moves harmonically in exactly the same way on the vertical diameter. But we assume that while *E* is at its leftmost position, *S* is in the middle of its harmonic oscillation, that is, at the centre of the circle. When *E* reaches the middle, *S* is at its highest position; when *E* arrives at the right-hand end, *S* is again in the middle; then *E* returns to the middle while *S* takes up the lowest position, and so on. We now ask the question, How does *S* move relative to the point *E*? That is to say, we imagine ourselves to be in *E* without knowing we are moving with *E*. How does *S* appear to move relative to us now? By considering the rectangles shown, and noticing that their diagonals are all of the same length, so that the diagonals of the rectangles in a quarter of the circle are all of the same length, we see that *S* describes a *circle* round *E* (in Figure 2 anticlockwise). And if we examine closely the inclination to the horizontal diameter of the connecting line *ES*, we realize too that this apparent circular motion of *S* about *E* is uniform. Obviously *E* likewise

moves with uniform circular motion with respect to *S*. Thus the fact that *E* and *S* execute harmonic motion in mutually perpendicular directions with the same amplitude and period might *perhaps* be used as a basis for the fact that "Seen from *E*, *S* moves in a circle round *E* with constant speed."

Now let us suppose that, in addition to its harmonic oscillation in the horizontal, the point *E* is simultaneously carrying out an oscillation in the vertical. For *this* oscillation let the maximium displacement from the middle position be *h* (Figure 3) and the time necessary for a complete oscillation be just *half as great* as for the complete horizontal oscillation. Figure 3 shows how the different positions point *E* takes up can very easily be constructed. As a result of the two simultaneous oscillations it carries out, *E* describes a figure-of-eight, a harmonic lemniscate as we shall call it (this is not Bernoulli's lemniscate).

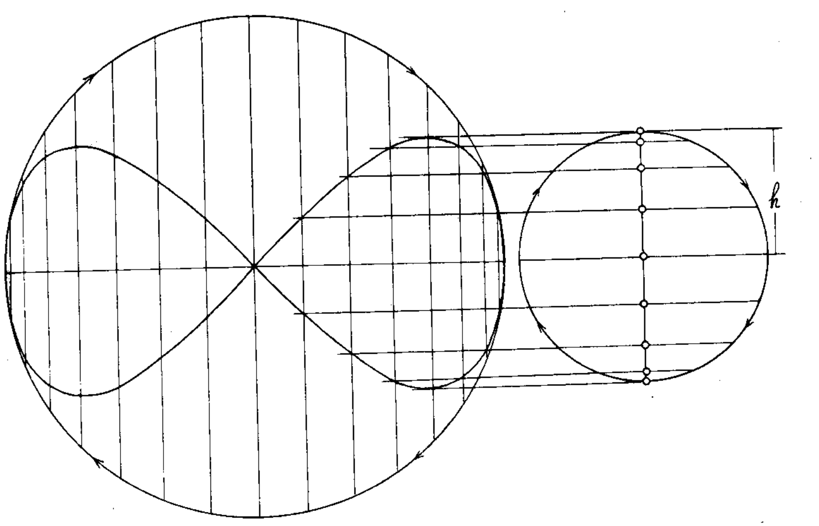



Fig. 3

Now imagine this harmonic lemniscate not in the plane of the figure but set at right-angles to it. Viewed perpendicularly to the plane of the figure it would thus appear to be the horizontal diameter. Suppose *E* moves along this figure-of-eight; and suppose that simultaneously *S* moves in a precisely similar figure-of-eight at right-angles to the plane of the figure, so that viewed perpendicularly it appears to be the vertical diameter. Fig. 4 illustrates this. When *E* is in positions 1, 2, 3..., *S* is at $\bar{1}, \bar{2}, \bar{3}, \ldots$ respectively. Since *corresponding* points such as 1 and $\bar{1}$ , 2 and $\bar{2}$ ,



$V^- = L^{-3}$ :

$$V^- = V_0^- \frac{1}{(1+at)^3} \qquad \textbf{(3.22)}$$

This equation may also be written as:

$$V^- = V_0^- \left( 1 - \frac{3at}{(1+at)^3} - \frac{3a^2t^2}{(1+at)^3} - \frac{a^3t^3}{(1+at)^3} \right) \qquad \textbf{(3.23)}$$

Figure 3.8 shows the expansion of a cube in counterspace. The various parts of this figure correspond to the terms in equation (3.23) in the following way : $V_0^- \frac{3at}{(1+at)^3}$ is the counterspatial volume which is occupied by the planes between the corners of the cubes, for instance the plane a between the points $A_1$, $A_2$, $B_2$, $D_2$ and $E_2$.

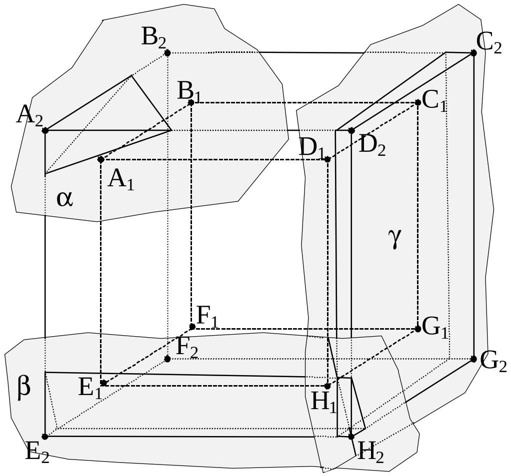



**Figure 3.8**

$V_0^- \frac{3a^2t^2}{(1+at)^3}$ is the counterspatial volume which is occupied by the planes between the edges of the cubes, for instance the plane b between the points $E_1$, $H_1$, $E_2$, $H_2$, $A_2$, $D_2$, $F_2$ and $G_2$.

$V_0^- \frac{a^3t^3}{(1+at)^3}$ is the counterspatial volume which is occupied by the planes between the sides of the cubes, for instance the plane g between the points $C_1$, $D_1$, $H_1$, $G_1$, $C_2$, $D_2$, $H_2$ and $G_1$.

Because the 'freedom of rotation' of the three representatively chosen planes diminishes from a to b to g , the figure demonstrates that the counterspatial volume between the corners of the cubes is much larger than the counterspatial volume between the edges of the cubes and that the latter is in turn much larger than the counterspatial volume between the sides of the cubes. Therefore, in equation (3.23), the terms $V_0^- \frac{a^3t^3}{(1+at)^3}$ and $V_0^- \frac{3a^2t^2}{(1+at)^3}$ may be ignored, in agreement with quadrilaterals *adbb´*:

$$A^-_{corners} = 4A^+_{adbb'} = 4\sqrt{2}\,x^+_{bo_2}\,\frac{1}{\sqrt{2}}\left(x^+_{ao_2} - x^+_{bo_2}\right)$$
$$= 4\frac{2}{L_0(1+at)}\left(\frac{2}{L_0} - \frac{2}{L_0(1+at)}\right) = \frac{16at}{L_0^2(1+at)^2} = A_0^-\,\frac{2at}{(1+at)^2} \qquad \textbf{(3.20)}$$

which is in agreement with the result of equation (3.12).

3.4 Thermal expansion in three dimensions: an outline

In this section, expansion in three dimensions is treated in outline. This is done without providing a sound mathematical basis, unlike the treatment of the expansion in two dimensions in the previous sections of part 3, which was based on the mathematical instrument developed in part 2.

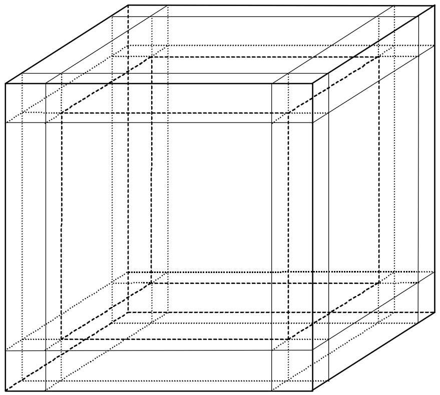
Figure 3.7 shows an expanding cube. The equation for expansion of a cube is:

$$V^+ = V_0^+\left(1 + 3at + 3a^2t^2 + a^3t^3\right) \qquad \textbf{(3.21)}$$

Here, $V_0^+$ is the original volume of the cube and $V^+$ the volume of the cube after expansion. The term $V_0^+ \cdot 3at$ corresponds to the volume of the six blocks on the sides of the original cube the term $V_0^+ \cdot 3a^2t^2$ , to the twelve beams on the edges and the term $V_0^+ \cdot a^3t^3$ to the eight little cubes at the corners.

The figure clearly demonstrates that $V_0^+ \cdot a^3t^3$ is an order of magnitude smaller than $V_0^+ \cdot 3a^2t^2$ and this in turn an order of magnitude smaller than $V_0^+ \cdot 3at$ . In equation (3.21), therefore, the terms $V_0^+ \cdot a^3t^3$ and $V_0^+ \cdot 3a^2t^2$ may be ignored.

**Figure 3.7**



The equation for expansion of a cube in counterspace can be obtained by raising the equation for linear thermal expansion in counterspace, equation (3.6), to the third power and substituting $V_0^- = L_0^{-3}$ and



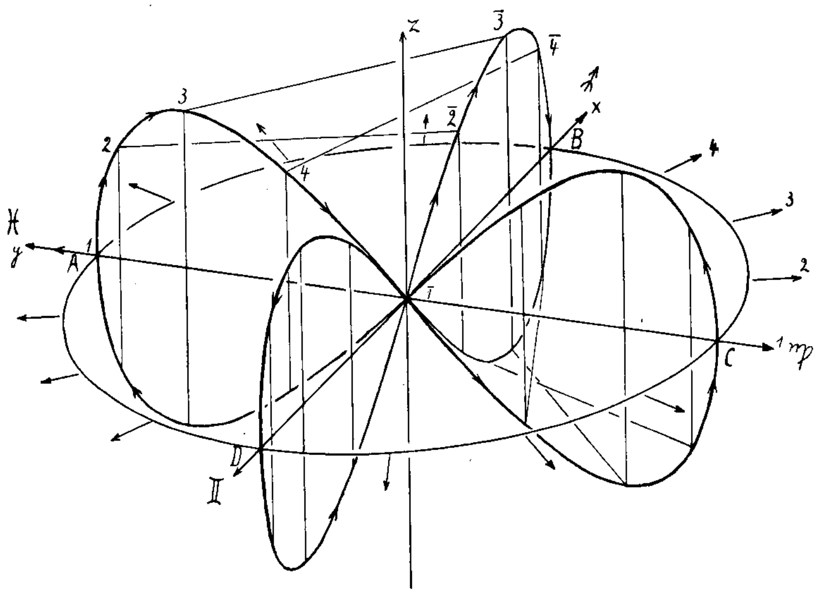



Fig. 4

3 and $\overline{3}$ , etc. *always have the same height above (or depth below) the plane ABCD* (this follows from the construction in Figure 3), *S* still moves uniformly in a circular orbit *relative to E*, as does *E* about *S*.

This gives us the model — to begin with extremely simplified — on which we are going to base the movements of Earth and Sun. Subjecting *both* Earth and Sun to a further movement modifies the harmonic lemniscates in such a way that they become spatially progressive. The relative circular motion of *S* about *E* shows itself in the passage of the Sun through the zodiac. The elevation and depression, *h*, above and below the central plane *ABCD* has no role in this apparent course, provided *h* is relatively small; this is well established empirically. Such loop-forms, at first seemingly fantastic, can thus be brought into complete agreement with outer observation. The basic form is reminiscent of the blood circulation.

II. Kepler discovered that, more precisely, *E* describes an ellipse relative to *S*, an ellipse with *S* at one of its foci, and that furthermore the connecting line *SE* sweeps out equal areas in equal times of this relative motion. But Kepler kept the Sun *S* at rest. With a modification of the above picture there is even agreement with these laws of Kepler's. One lemniscate must be made very slightly shorter than the other and be shifted a little from the other's middle point. The two deviations are barely

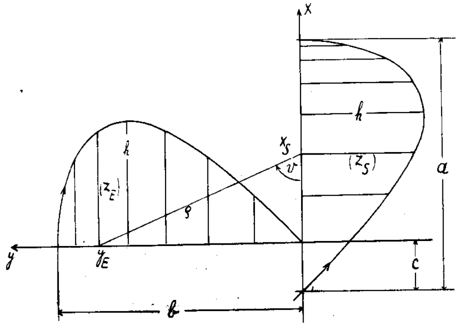



Fig. 5

noticeable at the scale of Figure 4. An additional slight variation (itself harmonic in nature) must be made to the harmonic oscillation. The precise details follow.

The motions of a point *S* in the *x-z* plane and of a point *E* in the *y-z* plane are given by

$$\left.\begin{aligned} x_S &= a\cos j - c \\ z_S &= h\sin 2j \end{aligned}\right\}(1) \qquad \left.\begin{aligned} y_E &= b\sin j \\ z_E &= h\sin 2j \end{aligned}\right\}(2)$$

where $a^2 = b^2 + c^2$ (and *c* is very small compared with *a* and *b*, in fact

$\frac{c}{a} = e = 0\cdot 0167$ , $\frac{b}{a} = \sqrt{1-e^2} = 0\cdot 99986$ ) , $j$ is a variable parameter whose

connection with time *t* will be given later and *h* a quantity which is certainly smaller than *a*. The equations in rectangular coordinates of the orbital curves are:

$$z_S = \frac{2h}{a^2}(x+c)\sqrt{a^2-(x+c)^2} \quad , \quad z_E = \frac{2h}{b^2}\cdot y\cdot\sqrt{b^2-y^2} \quad .$$

Figure 5 shows the curves rotated into the *x-y* plane. Here, for clarity, the relative size of *c* is greatly exaggerated. The elevation above or depression below the *x-y* plane (call it *z*) is proportional to the product of the deviations $x + c$ and $y$ of the individual motions from the middle positions:

$$z = \frac{2h}{ab}(x+c)y \quad .$$

Other approaches for *z* are possible, e.g. *z* proportional to the square root of $(x + c)y$.

For the distance *SE* we have $r = a - c\cos j$ . To find the relative motion



respect to the other ones. Of these two quadrilaterals, the outer one corresponds to the inner quadrangle and the other way round.

Now if one wants to determine the counterspatial area of the quadrangle *AA´C´C* – that is to say the area of the counterspace enveloping this quadrangle – all one needs to do is calculate the area of the quadrilateral *aa´c´c*. This follows from equation (2.8), which states that these two are equal.

From the polar-reciprocal relationship between the quadrangles and the quadrilaterals, the following can be derived for the *x*-co-ordinate of the point of intersection of *a* with $o_2$:

$$x^+_{ao_2} = \frac{1}{x^+_A} = \frac{2}{L_0} \qquad \textbf{(3.15)}$$

and for the *x*-co-ordinate of the point of intersection of *b* with $o_2$:

$$x^+_{bo_2} = \frac{1}{x^+_B} = \frac{2}{L_0(1+at)} \qquad \textbf{(3.16)}$$

By some simple arithmetic, we can immediately derive the area of the quadrangle *AA´C´C* in counterspace:

$$A^-_0 = A^+_{aa'c'c} = \left(\sqrt{2}\,x^+_{ao_2}\right)^2 = \left(\frac{2\sqrt{2}}{L_0}\right)^2 = \frac{8}{L_0^2} = \frac{8}{A^+_0} \qquad \textbf{(3.17)}$$

which is in agreement with the result of equation (3.9).

Similarly, the following holds for the counterspatial area of the quadrangle *BB´D´D*:

$$A^- = A^+_{bb'd'd} = \left(\sqrt{2}\,x^+_{bo_2}\right)^2 = \left(\frac{2\sqrt{2}}{L_0(1+at)}\right)^2 = \frac{8}{L_0^2(1+at)^2} = \frac{A^-_0}{(1+at)^2} \qquad \textbf{(3.18)}$$

which is in agreement with the result of equation (3.10).

The counterspatial area of four quadrangles *ACDB* along the sides corresponds to four small quadrilaterals *acdb*:

$$A^-_{sides} = 4A^+_{acdb} = 4\left(\frac{1}{\sqrt{2}}\left(x^+_{ao_2} - x^+_{bo_2}\right)\right)^2$$

$$= 2\left(\frac{2}{L_0} - \frac{2}{L_0(1+at)}\right)^2 = \frac{8a^2t^2}{L_0^2(1+at)^2} = A^-_0\,\frac{a^2t^2}{(1+at)^2} \qquad \textbf{(3.19)}$$

which is in agreement with the result of equation (3.11).

The counterspatial area of four quadrangles *ADBB´* at the corners corresponds to four

$$A^- = A_0^- - A^-_{corners} - A^-_{sides} = A_0^-\left(1 - \frac{2at}{(1+at)^2} - \frac{a^2t^2}{(1+at)^2}\right) \qquad \textbf{(3.13)}$$

This shows that in the expansion of a plane, the corners contribute most to the decrease of the total counterspatial area.

Equation (3.13) may also be written as:

$$A^- = A_0^- \frac{1}{(1+at)^2} \qquad \textbf{(3.14)}$$

This expression has the drawback that the contributions of the different portions of the expanding square can no longer be discriminated. However, it now becomes clear that the equation for expansion of a square in counterspace, equation (3.14), is equal to the square of the equation for linear thermal expansion in counterspace, equation (3.6). These equations are therefore in agreement with each other.

3.3 Thermal expansion in two dimensions: indirect calculation

The calculation of the counterspatial area of an expanding square may also be carried out in a different way. This is by using the result of section 2.2, that the area of a plane in counterspace equals the area of its polar-reciprocal inverse in the Euclidean space. The gist of this argument is that by reciprocation, the area which *envelops* a square is transformed into the *internal* space of the inverse of this square. This internal area may then be determined in the usual way in the Euclidean space. This indirect method is less suitable for building up an idea of what happens in counterspace upon expansion of an area, but it saves some arithmetic. In the following I shall work out the calculation according to the indirect method.

In figure 3.6, the expansion of a plane is shown once again. Because I now direct my attention to counterspace, that is to say the lines which envelop the expanding plane, this has been drawn as a quadr*angle*, and not as a quadri*lateral* (square) like in figure 3.2. Of both quadrangles in figure 3.6, the polar-reciprocal inverses with respect to an arbitrary circle have been drawn. These are the two concentric quadri*laterals* (squares) which are shown rotated by 45° around $O_3$ with

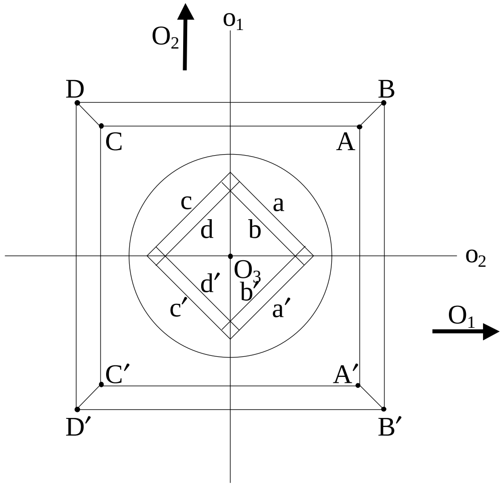



**Figure 3.6**



of $E$ with respect to $S$ we introduce the angle $n$ (see Figure 5). From $\tan n = \frac{y_E}{x_S}$ we get

$$\cos n = \frac{a\cos j - c}{a - c\cos j} \quad \text{or} \quad \cos j = \frac{a\cos n + c}{a + c\cos n} \quad .$$

Substituting for cos $j$ thus gives

$$r = \frac{b^2}{a + c\cos n} \quad .$$

$E$ thus moves in an ellipse with respect to $S$, $S$ being at a focus of the ellipse. For the element of area swept out by the relative motion of this radius vector $r$ we get

$$dA = \tfrac{1}{2} r^2 dn = \tfrac{1}{2} ab(1 - e\cos j)dj \quad ,$$

where $e = \frac{c}{a}$ . By Kepler’s second law this must be proportional to time, that is,

$dA = mdt$ ($m$ constant). The relationship between the quantity $j$ and time $t$ is thus given (taking $t$ = 0 when $j$ = 0) by Kepler’s equation:

$$j - e\sin j = \frac{2m}{ab}t = Bt \quad . \qquad (3)$$

Accordingly $x$ and $y$ do not vary precisely harmonically; but since $e$ is small, there is only a slight fluctuation, itself harmonic in nature.

Using equations (1), (2) and (3), after some transformations the accelerations in the $x$- and $y$-directions turn out to be:

$$\begin{aligned} \ddot{x}_S &= -a^3B^2\frac{x_S}{r^3} = -a^3B^2\frac{\cos n}{r^2} \ , \\ \ddot{y}_E &= -a^3B^2\frac{y_E}{r^3} = -a^3B^2\frac{\sin n}{r^2} \ . \end{aligned} \qquad (4)$$

Thus it is entirely possible to find movement forms such as those discussed in the preceding lectures which agree with Kepler’s laws. Showing that there is no contradiction with these laws was the aim of this conclusion.

What is the situation if we take the Gravitation Law as a basis? If a fixed coordinate system can be assumed then, according to this law, the following equations are true *without* the additional terms $P_S, P_E$ , … on the right:

$$
\begin{aligned}
m_S \cdot \ddot{x}_S &= G_x + P_S & m_E \cdot \ddot{x}_E &= -G_x + P_E \\
m_S \cdot \ddot{y}_S &= G_y + Q_S & m_E \cdot \ddot{y}_E &= -G_y + Q_E \qquad (5)^2 \\
m_S \cdot \ddot{z}_S &= G_z + R_S & m_E \cdot \ddot{z}_E &= -G_z + R_E
\end{aligned}
$$

where

$$G_x = k^2 m_E m_S (x_E - x_S) \frac{1}{r^3}$$

and $G_y$,$G_z$ are defined correspondingly.

The system (5) *without* the additional terms is true provided no other forces apart from gravity act on the masses $m_E$, $m_S$ being considered. But this is an assumption going beyond the Gravitation Law as an experimental law. Taking the possibility of other forces into account, we must — if we are to stay in the framework of classical mechanics — apply equations (5). By (4), the movements introduced above, namely

$$
\begin{aligned}
x_S &= a\cos j - c & x_E &= 0 \\
y_S &= 0 & y_E &= b \sin j \qquad j - e\sin j = Bt \\
z_S &= h \sin 2j & z_E &= z_S
\end{aligned}
$$

satisfy the equations:

$$
\begin{aligned}
\ddot{x}_S &= -a^3 B^2 \cdot x_S \cdot \frac{1}{r^3} & \ddot{x}_E &= 0 \\
\ddot{y}_S &= 0 & \ddot{y}_E &= -a^3 B^2 \cdot y_E \cdot \frac{1}{r^3} \\
\ddot{z}_S &= \frac{d^2}{dt^2}(h \sin 2j) & \ddot{z}_E &= \ddot{z}_S \quad .
\end{aligned}
$$

By comparison with (5), this corresponds to the following additional forces:

$$
\begin{aligned}
P_S &= 0 & P_E &= G_x \\
Q_S &= -G_y & Q_E &= 0 \qquad (*) \\
R_S &= m_S \cdot \frac{d^2}{dt^2}(h \sin 2j) & R_E &= \frac{m_E}{m_S} R_S \quad .
\end{aligned}
$$

* *Translator's note:* the values $P_S = 0$ and $Q_E = 0$ seem questionable as they imply $m_S = m_E$ . Any comments would be welcome!



sides of the square (figure 3.4) and one portion which contains the lines that envelop the corners of the square (figure 3.5).

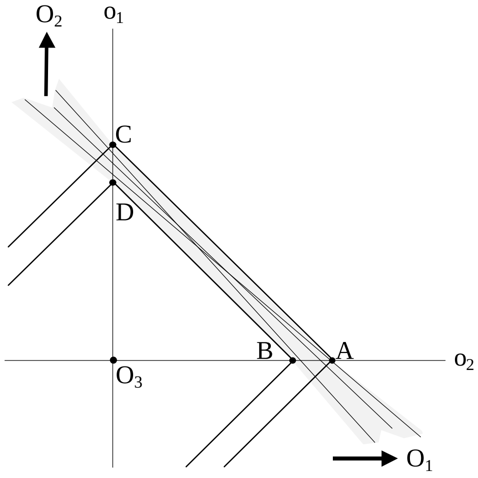



**Figure 3.4**

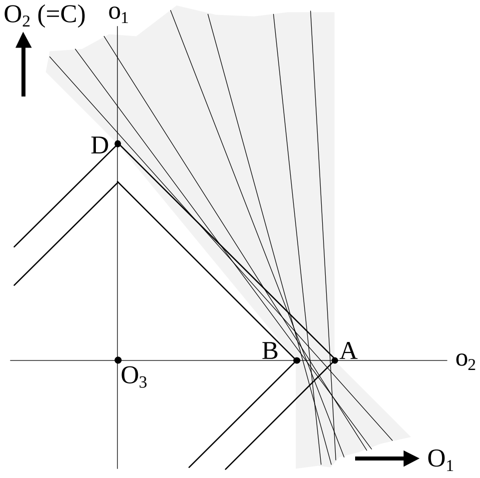



**Figure 3.5**

The counterspatial surface area of four of the quadrangles along the sides, such as the one in figure 3.4, can now be derived from equations (2.6), (2.4) and (3.9), with

$$x_B^+ = y_D^+ = \frac{L_0}{\sqrt{2}} \quad \text{and} \quad x_A^+ = y_C^+ = \frac{L_0(1+at)}{\sqrt{2}} \quad :$$

$$A^-_{sides} = 4\left(\frac{\sqrt{2}}{L_0} - \frac{\sqrt{2}}{L_0(1+at)}\right)^2 = \frac{8a^2t^2}{L_0^2(1+at)^2} = A_0^- \frac{a^2t^2}{(1+at)^2} \qquad \textbf{(3.11)}$$

In figure 3.5, the counterspatial area consisting of *half* of the lines enveloping one of the corners has been indicated. The counterspatial area corresponding to all four of the corners of the square is therefore *eight* times as large. Now the counterspatial area indicated in figure 3.5 can be derived from equations (2.6), (2.4) and (3.9), with

$$x_B^+ = \frac{L_0}{\sqrt{2}} \quad , \quad x_A^+ = y_D^+ = \frac{L_0(1+at)}{\sqrt{2}} \quad \text{and} \quad y_C^+ = \infty \quad :$$

$$A^-_{corners} = 8\left(\frac{\sqrt{2}}{L_0} - \frac{\sqrt{2}}{L_0(1+at)}\right)\left(\frac{\sqrt{2}}{L_0(1+at)} - \frac{1}{\infty}\right) = \frac{16at}{L_0^2(1+at)^2} = A_0^- \frac{2at}{(1+at)^2} \qquad \textbf{(3.12)}$$

Finally, from equations (3.11) and (3.12) it follows that:

In figure 3.3 the square under consideration has been turned by 45° around $O_3$, and only the portion of it that lies in the first quadrant is shown. The reason for this is that because of the symmetry, only the counterspatial area in the first quadrant needs to be considered; the total counterspatial area then equals four times this area. Furthermore, some lines of the counterspatial area in the first quadrant have been drawn.

The reason that in figure 3.3 the square is shown in a position turned by 45°, is that in this way the basic configuration of figure 2.3 may be recognised more easily.

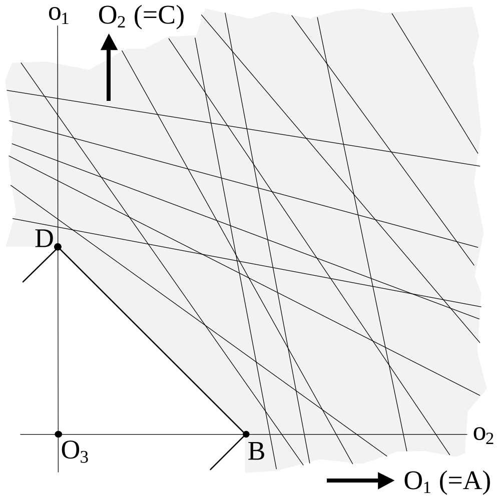



**Figure 3.3**

Points $B$ and $D$ of this basic configuration now lie at the corners of the square as drawn and points $A$ and $C$ coincide with $O_1$ and $O_2$, respectively, at infinity. The area in the first quadrant may now be calculated with equation (2.6) while also using (2.4). The negative area which corresponds to all four quadrants together is four times as large. With $x_B^+ = y_D^+ = \frac{L_0}{\sqrt{2}}$

$x_A^+ = y_C^+ = \infty$ , and $L_0^2 = A_0^+$ , it follows from (2.6) and (2.4) that:

$$A_0^- = 4\left(\frac{\sqrt{2}}{L_0} - \frac{1}{\infty}\right)^2 = \frac{8}{L_0^2} = \frac{8}{A_0^+} \qquad \textbf{(3.9)}$$

Analogously, using equation (3.9) at the last equality sign, the counterspatial area of the square after expansion is:

$$A^- = 4\left(\frac{\sqrt{2}}{L_0(1+at)} - \frac{1}{\infty}\right)^2 = \frac{8}{L_0^2(1+at)^2} = \frac{A_0^-}{(1+at)^2} \qquad \textbf{(3.10)}$$

From equation 3.10 it can be seen that $A^- < A_0^-$, meaning that the counterspatial area of the square decreases upon expansion.

Now I want to call your attention to the *difference* between the counterspatial areas of the square before and after expansion. The decrease in the counterspatial area may be discriminated in two portions: one portion which contains the lines that envelop the



(Note that $G_z = 0$ with our choice of coordinate system.)

Obviously we are not claiming that the forces and movements we are dealing with in reality are precisely these. The movements become still more complicated if the lemniscatory forms — which we can vary still further — are progressive rather than closed. It just has to be shown that it is never a matter of disregarding the thinking to date, but of developing it further.

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1. These are available from the Mathematisch-Astronomischen Sektion, Goetheanum, CH-4143, Dornach, Switzerland.
2. In our approach Newton's differential equations are satisfied by the *relative motions*.

# Thermal expansion in counterspace

by Ir. P.P. Veugelers
with a postscript by ir. R. van Romunde
English translation by drs. D. Sprangers

### *1 Introduction*

In the second lecture of his ‘Zweiter Naturwissenschaftlicher Kurs’[1], Rudolf Steiner speaks about the formula for the cubical expansion of solid substances:

$$V' = V\left(1 + 3at + 3a^2t^2 + a^3t^3\right) \tag{1.1}$$

In this formula, $a$ is the linear coefficient of expansion and $t$ the increase in temperature. It is common practice to ignore the last two terms of this equation, being powers of very small numbers. According to Steiner, however, one crosses out the most important thing while doing so, ‘das Allerwichtigste, worauf es ankommt, wenn man nun wirklich sachgemäß Wärmelehre treiben will.’ (‘the most important thing that matters, if one really wants to cultivate the theory of heat in keeping with the facts.’[2]) George Adams makes this remark understandable in his article ‘Wärmeausdehnung im Aetherraum’[3], by considering the expansion also as an event in the so-called ‘counterspace’[4]. Adams concludes his article by expressing the presumption that an arithmetical elaboration of his argument is hardly possible with the usual methods of calculation. I have made an attempt to expand the usual methods of calculation in such a way that an arithmetical treatment does become possible.

The arithmetical instrument for expansion in counterspace is developed in part 2. In doing so, I restrict myself to the two-dimensional space. The treatment in the three-dimensional space is analogous but too voluminous in the framework of this article to include it.

In part 3, I apply the arithmetical instrument to the expansion of a square. This chapter is concluded with a section in which the expansion of a cube is treated in broad outline, desisting from mathematical accuracy.

In part 4, finally, the results of part 3 are summarised and compared to Steiner's remark quoted above.

### *2 Measuring surface area in space and counterspace*

2.1 Determination of position in counterspace

In projective geometry, the position of a point on a line is unambiguously determined by three reference points on this line and a number which is called the cross ratio. If the three reference points are called $O_1$, $U$ and $O_2$,


**Figure 2.1**



and $d$ (shown in grey). In the same way, the term $A_0^+ \cdot a^2t^2$ corresponds to the four squares at the corners, for instance the one enclosed by the lines $a$, $b$, $c$ and $d$ (shown in dark grey).

Analogous to what was stated in the introduction, it is common practice to ignore the term $a^2t^2$ in equation (3.8) with respect to the term $2at$. This is justifiable because the former term (corresponding to the four squares at the corners) is an order of magnitude smaller than the latter (corresponding to the four rectangles along the sides). What matters now is to investigate whether it is still justifiable to ignore the former term if what happens in counterspace is taken into account.

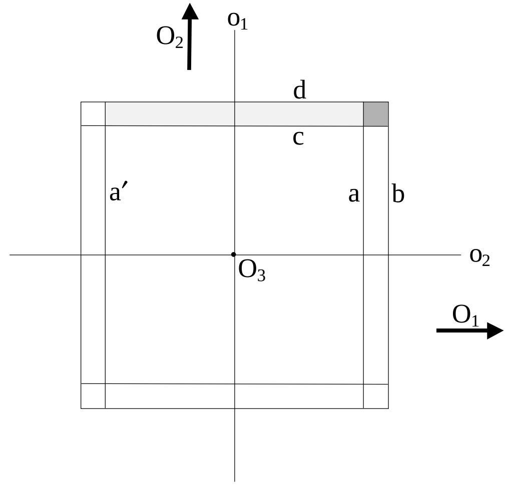


**Figure 3.2**

If you consider the square as an object in the Euclidean space, you direct your attention to the *points* which fill it up internally. If, however, you consider the square as an object in counterspace, you direct your attention to the *lines* which envelop it from infinity, from the periphery, inward.

Now what is the magnitude of the area, consisting of lines, which envelops the square under consideration? In section 2.2, an equation was derived for the area in counterspace of a quadrangle in the basic configuration of figure 2.3 (the quadrangle *BACD*). For any other quadrangle, the area in counterspace may now be determined by imagining it as being built up of a number of these basic configurations. This may be done in the same way as in the Euclidean space, where for instance the area of a trapezium is obtained by adding up a number of whole and half rectangles. In the following paragraphs, the area of the square in counterspace before and after expansion is calculated. The difference between the two may be discriminated in portions along the sides and portions around the corners. Of these subplanes, too, the counterspatial areas will be determined.

Now call the length before expansion in counterspace $L_0^-$. This is the part to the right of point $A$ in figure 3.1. Therefore:

$$L_0^- = x_A^- = \frac{1}{x_A^+} = \frac{1}{L_0^+} \tag{3.4}$$

Further, call the length after expansion in counterspace $L^-$. This is the part to the right of point $B$ in figure 3.1. Therefore:

$$L^- = x_B^- = \frac{1}{x_B^+} = \frac{1}{L^+} \tag{3.5}$$

By substituting equations (3.4) and (3.5) into (3.1), it follows that for linear thermal expansion in counterspace,

$$L^- = L_0^- \frac{1}{(1+at)} \tag{3.6}$$

This last equation may also be written as:

$$L^- = L_0^- \left(1 - \frac{at}{(1+at)}\right) \tag{3.7}$$

In this equation $L_0^- \dfrac{at}{(1+at)}$ is the length of the line segment between $A$ and $B$ in counterspace. (This result may be verified by subtracting equation (3.4) from (3.5) and substituting equation (3.1) into the result.)

Furthermore, from equation (3.7) it follows that $L^- < L_0^-$, in other words the length of the line segment in counterspace decreases upon expansion.

## 3.2 Thermal expansion in two dimensions

In figure 3.2 the expansion of a square plane is shown. The smaller square around $O_3$ is the area before expansion. The square surrounding it is the area after expansion.

If $A_0^+ = L_0^{+2}$ and $A^+ = L^{+2}$, then from equation (3.1) for linear expansion it follows that for expansion in two dimensions:

` $$A^+ = A_0^+\left(1 + 2at + a^2t^2\right) \tag{3.8}$$

In this equation, the term $A_0^+ \cdot 2at$ corresponds to the four rectangles along the sides of the square in figure 3.2, for instance the one enclosed by the lines $a$, $a'$, $c$



respectively, and the point to be determined is called $X$ (figure 2.1), then the cross ratio is defined as:

$$r \equiv (O1\,XUO2) \equiv \frac{O_1X}{XO2} \div \frac{O_1U}{UO2} \tag{2.1}$$

Here $O_1X$ is the length of the line segment between $O_1$ and $X$ if the line is traced from $O_1$ via $X$ to $O_2$. $XO_2$ is the length of the line segment between $X$ and $O_2$ in the same direction. $O_1U$ and $UO_2$ are defined similarly.

Now place $O_2$ at the infinity of the Euclidean space and $O_1$ at $O$, the origin of the Euclidean space. In this case the cross ratio passes into the Euclidean co-ordinate which I shall call $x^+$, because $XO_2 = UO_2 = \infty$ and therefore:

$$x^+ \equiv (OXU\infty) = \frac{OX}{OU} \tag{2.2}$$

with $OU$ being the unit length.

If, however, $O_2$ is placed at the infinity of *counterspace* (which is the origin of the Euclidean space, $O$) and $O_1$ at the origin of *counterspace* (which is the infinity of the Euclidean space), $XO_1 = UO_1 = \infty$. then The cross ratio now passes into the co-ordinate of counterspace which I shall call $x^-$:

$$x^- \equiv (\infty XUO) = \frac{OU}{OX} \tag{2.3}$$

Finally, from equations (2.2) and (2.3) it follows that:

$$x^- = \frac{1}{x^+} \tag{2.4}$$

so that for $x^-$ we may also use the term ‘reciprocal co-ordinate’.

## 2.2 Measuring surface area in counterspace

By reciprocating[5] the usual procedure for determining the magnitude of a surface area, an analogous procedure is generated for determining the magnitude of a surface area in counterspace.

In the text below, the left column shows the usual procedure for determining the magnitude of an area. The right column shows the polar-reciprocal inversion of this. The corresponding drawings, figures 2.2 and 2.3 respectively, are based on a Cartesian system of co-ordinates. This consists of an origin $O_3$, an $x$-axis $o_2$, a $y$-axis $o_1$ and the unit point $U$. Points $O_1$ and $O_2$, which are indicated by arrows, are the points of intersection of $o_2$ and $o_1$, respectively, with the line at infinity ($o_3$).
For clarity it should be noted that figure 2.3 is generated by reciprocating lines $a$, $b$, $c$ and $d$ and point $P$ in figure 2.2 with respect to the unit circle shown. This generates points $A$, $B$, $C$ and $D$ and line $p$. Also, $O_1$ is the inverse of $o_1$, $O_2$ the inverse of $o_2$ and $O_3$ the inverse of $o_3$.

Measuring surface area in Euclidean space

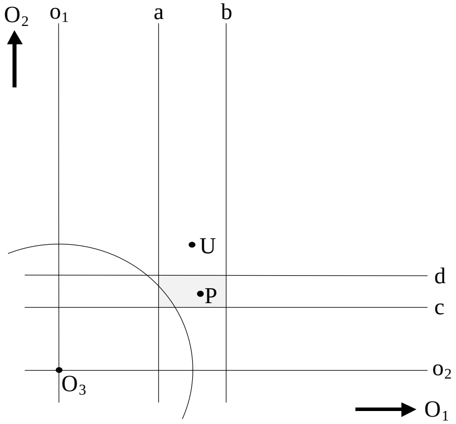


**Figure 2.2** The surface area to be determined (indicated in grey) is fixed by the lines $a$, $b$, $c$ and $d$ and an arbitrary point $P$ within this area. It consists of the set of points which are generated by translation of $P$ without crossing one of the lines $a$, $b$, $c$ and $d$.

The point of intersection of $a$, $b$ and $o_1$ is is point $O_2$ on $o_3$, the line at the infinity of the Euclidean space. ($a$, $b$ and $o_1$ are therefore called parallel.)

The point of intersection of $c$, $d$ and $o_2$ is point $O_1$ on $o_3$. ($c$, $d$ and $o_2$ are therefore called parallel.)

Measuring surface area in counterspace

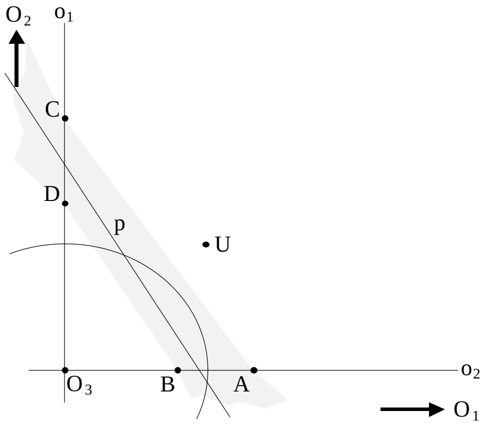


**Figure 2.3** The surface area to be determined (indicated in grey) is fixed by the points $A$, $B$, $C$ and $D$ and an arbitrary line $p$ within this area. It consists of the set of lines which are generated by rotation of p without passing one of the points $A$, $B$, $C$ and $D$.

The line connecting $A$, $B$ and $O_1$ is line $o_2$ through $O_3$, the point at the infinity of counterspace.

The line connecting $C$, $D$ and $O_2$ is line $o_1$ through $O_3$.



The magnitude of the surface area considered is determined by the product of the distance between lines $a$ and $b$ and the distance between lines $c$ and $d$ in the Euclidean space:

$$A^+ = \left(x^+_{o_2 b} - x^+_{o_2 a}\right)\left(y^+_{o_1 d} - y^+_{o_1 c}\right) \quad \textbf{(2.5)}$$ [6]

The magnitude of the surface area considered in counterspace is determined by the product of the distance between points $A$ and $B$ and the distance between points $C$ and $D$ in counterspace:

$$A^- = \left(x^-_B - x^-_A\right)\left(y^-_D - y^-_C\right) \qquad \textbf{(2.6)}$$

From the polar-reciprocal relationship between figures 2.2 and 2.3 it follows that:

$$x^+_{o_2 a} = \frac{1}{x^+_A} = x^-_A \;,\quad x^+_{o_2 b} = \frac{1}{x^+_B} = x^-_B \;,\quad y^+_{o_1 c} = \frac{1}{y^+_C} = y^-_C \;,\quad y^+_{o_1 d} = \frac{1}{y^+_D} = y^-_D$$

From this and equations (2.5) and (2.6), finally, it follows that: **(2.7)**

$$A^+ = A^- \qquad \textbf{(2.8)}$$

Thus, the magnitude of a surface area equals that of the polar-reciprocal inverse of this area in counterspace. This result is in agreement with the fact that by reciprocation, each point within an area is transformed one-to-one into a line.

## *3 Thermal expansion in space and counterspace*

### 3.1 Thermal expansion in one dimension

In figure 3.1, the expansion of a line segment has been shown in a Euclidean system of co-ordinates. $OA$ is the line segment before expansion

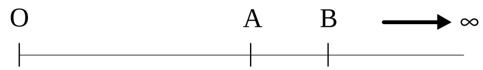


**Figure 3.1**

and has a length $L_0^+$, and $OB$ isthe line segment after expansion and has a length $L^+$. According to the equation for linear thermal expansion, it can now be stated that:

$$L^+ = L_0^+(1 + at) \qquad \textbf{n(3.1)}$$

In this equation , $L_0^+ at$ is the length of the line segment between $A$ and $B$.

An equation for linear thermal expansion in counterspace can be derived as follows.
We have:

$$L_0^+ = x_A^+ \qquad \textbf{(3.2)}$$

and

$$L^+ = x_B^+ \qquad \textbf{(3.3)}$$